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16. ABSTRACT

Automobiles are increasingly equipped with autonomous and semi-autonomous technologies such as adaptive cruise control and automated lane-keeping. It is apparent that increasing numbers of these smart vehicles will have a dramatic impact on network-level mobility factors such as traffic congestion and travel times. By enabling platooning of groups of smart vehicles along the roadway, these autonomous capabilities can improve mobility. A platoon is two or more vehicles which are able to maintain short headways between them using, e.g., adaptive cruise control (ACC) (which allows a vehicle to use radar or Light Detection and Ranging (LIDAR) to automatically maintain a specified distance to the preceding vehicle) or cooperative adaptive cruise control (CACC). This report describes the study of road capacity models and vehicle routing behavior in transportation networks with mixed autonomy. That is, networks in which a fraction of the vehicles on each road are equipped with autonomous capabilities, such as adaptive cruise control, that enable vehicle reduced headways and increased road capacity. In this research, a mixed traffic profile was considered, where a fraction of vehicles are smart and able to form platoons, and the remaining are regular vehicles that are manually driven. Two models were developed for road capacity under mixed autonomy that are based on the fundamental behavior of autonomous technologies such as adaptive cruise control. This study included simulation and modeling of transportation networks in which the delay on each road or link is an affine function of two quantities: the number of vehicles with autonomous capabilities on the link and the number of regular vehicles on the link. The microscopic traffic simulator "Simulation of Urban MObility" (SUMO) was used to validate the models developed in this study.

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Final Report: Control and Management of Urban Traffic Networks with Mixed Autonomy

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Abstract

We study capacity models and routing behavior in transportation networks with mixed autonomy, that is, networks in which a fraction of the vehicles on each road are equipped with autonomous capabilities such as adaptive cruise control that enable reduced headways and increased road capacity. In this report, we consider a mixed traffic profile where a fraction of vehicles are smart and able to form platoons, and the remaining are regular and manually driven. We develop two models for road capacity under mixed autonomy that are based on the fundamental behavior of autonomous technologies such as adaptive cruise control. We then consider transportation networks in which the delay on each road or link is an affine function of two quantities: the number of vehicles with autonomous capabilities on the link and the number of regular vehicles on the link.

We particularly study the price of anarchy for such networks, that is, the ratio of the total delay experienced by selfish routing to the socially optimal routing policy. Unlike the case when all vehicles are of the same type, for which the price of anarchy is known to be bounded, we first show that the price of anarchy can be arbitrarily large for such mixed autonomous networks. Next, we define a notion of asymmetry corresponding to the maximum possible travel time improvement due to the presence of autonomous vehicles. We show that when the degree of asymmetry of all links in the network is bounded by a factor less than 4, the price of anarchy is bounded. We also bound the bicriteria, which is a bound on the cost of selfishly routing traffic compared to the cost of optimally routing additional traffic. These bounds depend on the degree of asymmetry and recover classical bounds on the price of anarchy and bicriteria in the case when no asymmetry exists. Further, we show with examples that these bound are tight in particular cases. Finally, we detail some simulation studies validating two mixed autonomy road capacity models, and provide some basic knowledge of the microscopic traffic simulator SUMO used to validate the models.

1 INTRODUCTION

Automobiles are increasingly equipped with autonomous and semi-autonomous technologies such as adaptive cruise control and automated lane-keeping. These technologies are often marketed to consumers as safety or convenience features, but it is apparent that increasing numbers of these *smart* vehicles will have dramatic impact on network-level mobility factors such as traffic congestion and travel times [1]. A primary mechanism whereby such autonomous capabilities can improve mobility is by enabling *platooning* of groups of smart vehicles along the roadway. A platoon consists of two or more vehicles which are able to automatically maintain short headways between them using, *e.g.*, adaptive cruise control (ACC), which allows a vehicle to use radar or LIDAR to automatically maintain a specified distance to the preceding vehicle, or cooperative adaptive cruise control (CACC) which augments ACC with vehicle-to-vehicle communication.

When all vehicles in the system are smart, platooning has the potential to increase network capacity by as much as three-fold [2]. Platooning can help to smooth traffic flow and avoid *shockwaves* of slowing vehicles [3, 4, 5, 6, 7, 8], and at signalized intersections, platoons can synchronously accelerate at green lights [2, 9]. However, in a *mixed* autonomy setting—where only a fraction of the vehicles are smart and the remainder are *regular*, human-driven vehicles—the benefits of platooning are less clear. On freeways, simulation results suggest that high penetration rates of smart vehicles are required to realize significant improvement in traffic flow [10, 11, 12, 13, 14, 15].

In this work, we develop capacity models for roads with mixed autonomy in order to study routing behavior on transportation networks. We make the assumption that the additional travel time caused by congestion on a road is inversely related to capacity and proportional to the total number of vehicles on the road. Given a network of roads leading from origins to destinations, selfish vehicles will choose the route that minimizes total delay, achieving a Wardrop equilibrium [16, 17]. It has long been known that a Wardrop equilibrium may be suboptimal in the sense that a social planner is able to prescribe routes that achieve a lower total delay for all vehicles in the network. The ratio of the socially optimal delay to the worst possible Wardrop equilibrium is called the *price of anarchy* [18, 19]. For affine separable cost functions, when only one type of vehicle is present (*i.e.*, no smart vehicles), it is known that the price of anarchy cannot exceed 4/3 [20].

In a mixed autonomy setting, however, a social planner is able to route smart vehicles differently than regular vehicles to maximize capacity. In this paper, we first show that this increased flexibility leads, remarkably, to an unbounded price of anarchy. Next, we make the assumption that the possible travel time improvement due to the presence of autonomous vehicles is bounded by a factor $k < 4$. We call this factor the *degree of asymmetry* of the network. Under this assumption, we prove that the price of anarchy cannot exceed $\frac{4}{4-k}$, which recovers the classical bound when $k = 1$, *i.e.*, the case when smart vehicles do not enable any improvement in travel time. We show via examples that this bound is tight for $k = 1$ and $k = 2$.

Next, we provide a bound on the cost of selfish routing relative to the cost of optimally routing additional traffic, called the bicriteria bound [20], [21]. We prove that traffic at a Wardrop Equilibrium will not exceed the cost of optimally routing $1 + \frac{k}{4}$ as much traffic of each type, where k is the degree of asymmetry in the network. We demonstrate by example that the bicriteria bound is tight for $k = 4$. Similar to the price of anarchy, when the asymmetry is unbounded we show that the bicriteria is unbounded as well. This runs counter to the case of single-type traffic where the bicriteria is bounded by 2 for any separable continuous and nondecreasing cost function in which the delay on a road depends only on the traffic on that road [20].

2 PREVIOUS WORKS

In this section, we address related models in the literature and highlight the difference between these and our model. Due to the breadth of the field, we give a limited overview of the literature on the price of anarchy – see [22] for a broader survey of literature related to Wardrop equilibria and [23] for a wider background on the price of anarchy in transportation networks. For definitions of the terms used in this section see Section 4.3.

Roughgarden and Tardos [20] bound price of anarchy and bicriteria for separable monotonic cost functions, and Roughgarden [24] gives a more general method for determining price of anarchy in the separable case.

Chau and Sim [25] bound the price of anarchy for symmetric cost operators with convex social cost for both nonelastic and elastic demands. Perakis discusses nonseparable, asymmetric, nonlinear costs in [26], though only for monotone latencies *i.e.* satisfying the property

$$\langle c(z) - c(v), z - v \rangle \geq 0 \ , \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors.

Correa et. al [27] present a unified framework for deriving price of anarchy and bicriteria for nonseparable monotone functions. Sekar et. al [28] analyze the price of anarchy when users have different beliefs about the delay on a road, but experience the same actual cost, dictated by a monotone cost function.

Unlike these previous works, we present a price of anarchy and bicriteria bound for a class of *nonmonotone* and pairwise separable affine cost functions. We show that our bound simplifies to the classic bounds for affine monotone cost functions in [20], [25], and [27] when there is no asymmetry in how the vehicle types affect congestion.

3 Modeling Mixed Autonomy

The Highway Capacity Manual defines the capacity of a road as the maximum possible flow rate on the road in vehicles per hour [29]. Capacity is primarily limited by the average headway that vehicles maintain while traveling on the road, and the HCM recommends a nominal saturation flow rate of 1900 vehicles per hour (vph) per lane to capture typical behavior of drivers, which corresponds to a headway of $3600/1900 = 1.89$ seconds (s).

With the emergence of semiautonomous driving technology such as cooperative adaptive cruise control (CACC), it is projected that the headway can be reduced to approximately 0.8 s, which corresponds to a road capacity of 4500 vph [30]. However, to achieve this nearly 2.5-fold increase in capacity requires every vehicle to maintain shorter headways. What happens when only a fraction of vehicles are equipped with the required technology to achieve reduced headway? In this section, we consider two possible models for reduced headways and increased capacity in this *mixed autonomy* setting. We say a vehicle is *smart* if it is equipped with semiautonomous driving capabilities that enable reduced headways, and *regular* otherwise.

3.1 Capacity Model 1

We first consider a scenario in which each smart vehicle is able to maintain reduced headway with any preceding vehicle, regardless of whether that vehicle is also equipped with driver assistance technology. This scenario is plausible when smart vehicles are able to accurately localize any surrounding vehicle without communication. At the moment, it is unclear whether a sufficiently sophisticated sensor suite for this task will be omnipresent on autonomous vehicles. Indeed, the sensors likely required to achieve this feat, such as lidar, are currently prohibitively expensive for mass adoption. On the other hand, companies such as Tesla have focused on enabling autonomous technology by relying primarily on cameras [31].

Suppose that m is the capacity of the road when fully utilized by regular vehicles, and M is the capacity when fully utilized by smart vehicles, where $M > m$. Let α be the *average* fraction of smart vehicles on the road. We call α the *autonomy level* of the road. Define $C(\alpha)$ to be the capacity of the road under autonomy level α. We propose the following approximation:

$$C(\alpha) = (\alpha M^{-1} + (1-\alpha)m^{-1})^{-1}. \tag{2}$$

Here is a justification for (2). First, assume that every smart vehicle follows the preceding vehicle (whether it is smart or regular) with time gap of $t_2 = M^{-1}$. Second, assume that each regular vehicle follows the preceding vehicle with time gap of $t_1 = m^{-1}$ where $t_1 > t_2$. Finally, since a fraction α of vehicles are smart, the effective headway of the road with autonomy level α is $\alpha t_2 + (1-\alpha)t_1$, which implies (2). Note that this approximation is valid regardless of how the smart vehicles are distributed among the regular vehicles, so long as the autonomy level is α.

In Figure 1, we plot road capacity as a function of the fraction of smart vehicles α, assuming that the capacity when there are no smart vehicles ($\alpha = 0$) is 1900 vph and when there are only smart vehicles ($\alpha = 1$) is 4500 vph.

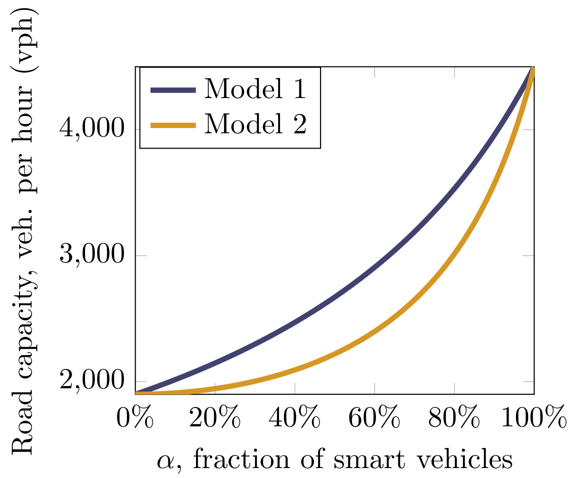

Figure 1: Road capacity as a function of the autonomy level α, *i.e.*, the ratio of smart vehicles on the road. In Model 1 (dark/blue), it is assumed that smart vehicles are able to maintain shorter headways with any preceding vehicle. In Model 2 (light/orange), it is assumed that smart vehicles are only able to maintain a shorter headway if the preceding vehicle is also smart, with results in decreased capacity.

3.2 Capacity Model 2

We now consider an alternative model in which a smart vehicle is only able to maintain a short headway if the preceding vehicle is also smart. As above, suppose that m (respectively, M) is the capacity of the road when fully utilized by regular (respectively, smart) vehicles, and again let α be the average fraction of smart vehicles on the road.

Unlike the previous model, the capacity of the road now depends on the distribution of smart vehicles among regular vehicles. For example, consider a single-lane road with n vehicles, and suppose $n/2$ are regular and $n/2$ are smart. In the extreme case that the smart and regular vehicles are perfectly interleaved such that every smart vehicle is preceded by a regular one, then the capacity is m and no gain is achieved. In the other extreme case that a platoon of $n/2$ smart vehicles precedes $n/2$ regular vehicles, the throughput of the road becomes the same as (2). Therefore, a proper definition of the capacity of the road depends on the *stochastic process* of the vehicles traversing the road.

Here, we propose to model vehicle type as a Bernoulli process, *i.e.*, each vehicle is smart with probability α and regular with probability $1 - \alpha$ independently. In this case, the capacity of the road is approximated as

$$C(\alpha) = (\alpha^2 M^{-1} + (1-\alpha^2)m^{-1})^{-1}. \tag{3}$$

To derive (3), note that the time gap between two vehicles is $t_2 = M^{-1}$ if they are both smart and $t_1 = m^{-1}$ otherwise. Thus, one needs to count the fraction of pairs of smart vehicles, which is α^2. This implies that the average headway is $\alpha^2 t_2 + (1-\alpha^2)t_1$. Figure 1 also plots the capacity model (3).

4 MOTIVATION AND MATHEMATICAL FORMULATION

In this section we motivate our cost function for traffic in mixed autonomy. In Section 4.1, we show that the price of anarchy and bicriteria are unbounded for congestion games with affine cost functions in mixed autonomy, described in Section 4.2. Prompted by our negative result, in Section 4.3 we describe a pairwise separable cost function that is parameterized by the degree of asymmetry, as well as a more general class of nonseparable cost function.

4.1 Motivation

We provide a brief example of unbounded price of anarchy and bicriteria for congestion games under mixed autonomy. This example is similar in design to Pigou's example, as in [20], [32], and [23].

Example 1. *Consider the traffic network in Fig. 2, in which $\frac{1}{\zeta}$ unit of regular traffic and* 1 *unit of smart traffic need to travel from node s to node t, where $\zeta \geq 1$. The cost on road i, or the delay a car experiences from traveling on that road, is denoted $c_i(x, y)$.*

The routing with all traffic on the bottom road is at Wardrop Equilibrium, with a price of $C^{EQ} = (\frac{1}{\zeta}\zeta)(\frac{1}{\zeta}+1) = \frac{1}{\zeta}+1$. The optimal routing has the regular traffic on the top and smart traffic on the bottom with a cost of $C^{OPT} = \frac{1}{\zeta}\cdot 1 + \frac{1}{2}\cdot 0 = \frac{1}{\zeta}$. This results in a price of anarchy of $\zeta + 1$.

For the bicriteria, consider a situation in which we have a mass of $\frac{a}{\zeta}$ units of smart cars and a units of regular cars to route. We want to find the a that corresponds to, under optimal routing,

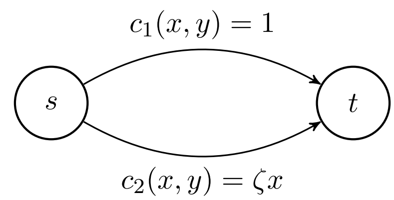

Figure 2: Example of a road network with price of anarchy and bicriteria that grow unboundedly with ζ.

a cost equaling that of C^{EQ} above. The optimal routing will have cost $\frac{a}{\zeta}$, which equals $\frac{1}{\zeta} + 1$ when $a = \zeta + 1$.

Here we see that both the price of anarchy and the bicriteria bound grow unboundedly with ζ. Due to this result, we state the following proposition:

Proposition 1. *The price of anarchy and bicriteria are unbounded for networks of mixed autonomy with pairwise separable affine functions.*

Because of this negative result, to provide a bounded price of anarchy and bicriteria in mixed autonomy we develop a class of cost functions with bounded asymmetry.

4.2 Affine Congestion Game Overview

Consider a network of n roads, with m origin-destination pairs, each with an associated mass of regular vehicles of volume α_i and mass of smart vehicles of volume β_i. Since we are considering a nonatomic congestion game, each user controls an infinitesimally small portion of that mass. We denote $\mathcal{X}$ as the set of feasible strategies which result in the entirety of each mass being routed from its origin to its destination, without violating conservation of flow (see [33] for a more detailed explanation).

The vector of all flows on the n roads is denoted by

$$z = \begin{bmatrix} x_1 & y_1 & x_2 & y_2 & \dots & x_n & y_n \end{bmatrix}^T ,$$

where x_i and y_i represent the mass of regular and smart vehicles, respectively, on road i. In this paper, we consider affine cost functions, meaning the cost on the roads resulting from a routing

$z \in \mathcal{X}$ can be written as

$$c(z) = Az + b \ ,$$

where $A \in \mathbb{R}^{2n}_{\geq 0}$ and $b = \begin{bmatrix} b_1 & b_1 & b_2 & b_2 & \dots & b_n & b_n \end{bmatrix}^T$. This yields social cost

$$C(z) = \langle c(z), z \rangle = (Az + b)^T z \ ,$$

and the social cost at optimal routing is then $C^{OPT} = \inf_{z \in \mathcal{X}} C(z)$. Vector b contains the constant terms; matrix A is the Jacobian of the road cost operator, and is not in general positive semidefinite, so the optimization is not convex.

4.3 Separability and Monotonicity

Having described the basic structure of the congestion game with affine costs, we describe the separability and monotonicity of our model. To do so, we define three notions of separability.

Definition 1. *A cost function $c(z) = Az + b$ is* separable *if A is a diagonal matrix.*

Definition 2. *A cost function $c(z) = Az + b$ is* pairwise separable *if A is a blockwise diagonal matrix with 2x2 blocks.*

Definition 3. *A cost function is* nonseparable *if it is neither separable nor pairwise separable.*

It is clear that separable costs do not model mixed autonomy if regular and smart cars affect delay differently but experience it identically. The slightly more general class of pairwise separable costs does provide a useful model, which we motivate as follow, using a capacity model similar to those in [9], [34], [35]:

Consider a road with regular car flow x and smart car flow y and autonomy level $\alpha = \frac{y}{x+y}$. We use Capacity model 1 in (2), and propose a road cost function in which delay is an affine function of vehicles on the road:

$$c(x, y) = b + r\frac{x + y}{C(\alpha)} = b + rM^{-1}y + rm^{-1}x.$$

Here b represents the time it takes to traverse a road in free-flow traffic and r determines how congestion scales as road utilization increases with respect to capacity.

To capture a notion of asymmetry in how the types of vehicles affect traffic, we can rewrite the cost on road i as follows:

$$c_i(x_i, y_i) = b_i + k_i a_i x_i + a_i y_i \ . \tag{4}$$

This leads to a cost function of the following form:

$$c(z) = Az + b = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix} z + b$$

where A is a block-diagonal matrix with blocks $A_i = \begin{bmatrix} k_i a_i & a_i \\ k_i a_i & a_i \end{bmatrix}$.

The parameter k_i allows us to represent the degree of asymmetry between the effect of regular and smart traffic on congestion on a specific road. Since in [35], [30], and [2], we see that vehicles not in a platoon require approximately 2.5 times more headway than vehicles in a platoon, we allow k_i to differ between roads, but generally expect it to be in the range $k_i \in [1, 4]$.

We find it useful to parameterize a class of cost functions by its maximum degree of asymmetry, as follows:

Definition 4. *Let $\mathcal{C}_k$ denote the class of pairwise separable cost functions for which* $\max(k_i, \frac{1}{k_i}) \leq k$ $\forall i$ *for some constant k. We call k the* maximum degree of asymmetry *of this class of cost functions.*

In the more general model explored in Section 5.3, the delay on one road may depend on the flows on other roads. For example, if one road is fully congested, the roads feeding it will have additional delay. If this is the case, then (4) does not hold and the matrix A is not of block-diagonal form. In Section 5.3 we provide a bound for this model under certain conditions.

Throughout this paper, we deal with cost functions that satisfy element-wise monotonicity, defined as follows:

Definition 5. *A class of cost functions $\mathcal{C}$ is* elementwise monotone *if for all cost functions $c(v)$ drawn from $\mathcal{C}$, $\frac{\partial c_j}{\partial v_i}(v) \geq 0 \ \forall i, j$.*

In other words, a cost function is element-wise monotonic if increasing any flow of vehicles will not decrease the delay on any road. This will be the case for a class of cost functions of the form $c(z) = Az + b$ for which A has only nonnegative entries. Note that this is different from the general notion of monotonicity described in Section 2[1].

5 BOUNDING THE PRICE OF ANARCHY

In this section we present bounds for the price of anarchy and bicriteria. We proceed along the lines proposed in [27], reviewing that work in Section 5.1 and highlighting the differences that arise for a nonmonotone cost function. We then derive our bounds for nonmonotone pairwise separable costs in Section 5.2, and give a bound for nonseparable costs in Section 5.3.

5.1 Preliminaries

Smith [36] shows that any flow z^{EQ} at Wardrop equilibrium – in which all users sharing an origin and destination use paths of equal cost and no unused path has a smaller cost – satisfies the variational inequality for any feasible flow z:

$$\langle c(z^{EQ}), z^{EQ} - z \rangle \leq 0 \ . \tag{5}$$

A simple proof of this is provided in [33].

Correa et. al. [27] use this result to develop a general tool for finding price of anarchy and

[1]Our case is not in general monotone: consider a single road with $c(z) = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} z$, with $x = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $y = \begin{bmatrix} 0 & 2 \end{bmatrix}^T$. This results in $\langle c(x) - c(y), x - y \rangle = -1$.

bicriteria. To that end, they introduce the following parameters:

$$\beta(c,v) := \max_{z\in\mathbb{R}^{2n}_{\geq 0}} \frac{\langle c(v)-c(z), z\rangle}{\langle c(v), v\rangle} ,$$

$$\beta(\mathcal{C}) := \sup_{c\in\mathcal{C}, v\in\mathcal{X}} \beta(c,v) ,$$

where 0/0=0 by definition, $\mathcal{C}$ represents a class of cost functions, and $\mathcal{X}$ is the set of feasible routings.

In the following theorem, we adapt Correa *et. al*'s Theorem 4.2 [27] to when the cost function is not monotone. In the nonmonotone case $\beta(\mathcal{C})$ can be greater than 1, leading to an unbounded price of anarchy. For completeness, we overview the proof of Theorem 4.2 in [27].

Theorem 1. *Let z^{EQ} be an equilibrium of a nonatomic congestion game with cost functions drawn from a class $\mathcal{C}$ of nonseparable nonmonotone but elementwise monotone cost functions.*

(a) If z^{OPT} is a social optimum for this game, and $\beta(\mathcal{C}) < 1$, then $C(z^{EQ}) \leq (1-\beta(\mathcal{C}))^{-1}C(z^{OPT})$.

(b) If w^{OPT} is a social optimum for the same game with $1+\beta(\mathcal{C})$ times as many players of each type, then $C(z^{EQ}) \leq C(w^{OPT})$.

Proof. To prove part (a),

$$\begin{aligned}\langle c(z^{EQ}), z\rangle &= \langle c(z), z\rangle + \langle c(z^{EQ}) - c(z), z\rangle \\ &\leq \langle c(z), z\rangle + \beta(c, z^{EQ})\langle c(z^{EQ}), z^{EQ}\rangle \\ &\leq C(z) + \beta(\mathcal{C})C(z^{EQ}) \end{aligned} \tag{6}$$

and $C(z^{EQ}) \leq \langle c(z^{EQ}), z\rangle$. Completing the proof requires that $\beta(\mathcal{C}) \leq 1$.

To prove part (b), element-wise monotonicity implies the feasibility of $(1+\beta(\mathcal{C}))^{-1}w^{OPT}$, and using (5),

$$\langle c(z^{EQ}), z^{EQ}\rangle \leq \langle c(z^{EQ}), (1+\beta(\mathcal{C}))^{-1}w^{OPT}\rangle . \tag{7}$$

Then,

$$
\begin{aligned}
C(z^{EQ}) &= (1+\beta(\mathcal{C}))\langle c(z^{EQ}), z^{EQ}\rangle \\
&\quad - \beta(\mathcal{C})\langle c(z^{EQ}), z^{EQ}\rangle && (8) \\
&\leq (1+\beta(\mathcal{C}))\langle c(z^{EQ}), (1+\beta(\mathcal{C}))^{-1} w^{OPT}\rangle \\
&\quad - \beta(\mathcal{C})\langle c(z^{EQ}), z^{EQ}\rangle && (9) \\
&\leq C(w^{OPT}) \;, && (10)
\end{aligned}
$$

where (9) uses (7) and (10) uses (6). □

5.2 Pairwise Separable Costs

We now present a bound for the price of anarchy and bicriteria for the pairwise separable affine cost function when k, the maximum degree of asymmetry of the cost function, is bounded. In particular, when $k < 4$, the price of anarchy is bounded, and the bicriteria is bounded for any k. This is formalized as follows:

Theorem 2. *Let z^{EQ} be an equilibrium of a nonatomic congestion game with cost functions drawn from a class $\mathcal{C}_k$ of affine, pairwise separable, nonmonotone, elementwise monotone cost functions, where k parameterizes the maximum degree of asymmetry in the cost functions.*

(a) If z^{OPT} is a social optimum for this game, and $k < 4$, then $C(z^{EQ}) \leq \frac{4}{4-k} C(z^{OPT})$.

(b) If w^{OPT} is a social optimum for the same game with $1 + \frac{k}{4}$ times as many players of each type, then $C(z^{EQ}) \leq C(w^{OPT})$.

Proof. To prove this, we will show $\beta(\mathcal{C}_k) \leq \frac{k}{4}$ and then apply Theorem 1. For ease of notation, let $z^{EQ} \triangleq \begin{bmatrix} x_1^* & y_1^* & x_2^* & y_2^* & \dots & x_n^* & y_n^* \end{bmatrix}$.

Without loss of generality, and with a slight abuse of notation, we order the roads such that for $1 \leq i \leq \ell$, $c_i(x_i, y_i) = k_i a_i x_i + a_i y_i$ and for roads $\ell < i \leq n$, $c_i(x_i, y_i) = a_i x_i + k_i a_i y_i$, where $k_i \geq 1$.

Then,

$$\begin{aligned}\beta(c, z^*) &= \max_{z \in \mathbb{R}^{2n}_{\geq 0}} \frac{\langle c(z^*) - c(z), z\rangle}{\langle c(z^*), z^*\rangle} \\ &\leq \frac{\max_{z \in \mathbb{R}^{2n}_{\geq 0}} \langle A(z^* - z), z\rangle}{\langle Az^*, z^*\rangle} \\ &= \frac{\sum_{i=1}^{\ell} a_i \max_{x_i, y_i \geq 0}(k_i(x_i^* - x_i) + (y_i^* - y_i))(x_i + y_i)}{\langle Az^*, z^*\rangle} \\ &+ \frac{\sum_{i=\ell+1}^{n} a_i \max_{x_i, y_i \geq 0}((x_i^* - x_i) + k_i(y_i^* - y_i))(x_i + y_i)}{\langle Az^*, z^*\rangle}. \end{aligned} \tag{11}$$

We will bound the first term in (11), and the same can be done for the second term as well. Denote the inner term γ, so $\gamma(x_i, y_i) = (k_i(x_i^* - x_i) + (y_i^* - y_i))(x_i + y_i)$. This term is not concave, but is concave with respect to both x_i and y_i individually. Then, we use $f(x_i)$ to denote the function that maximizes γ with respect to y_i by solving $\frac{\partial \gamma}{\partial y_i}(x_i, y_i) = 0$, and $g(y_i)$ to denote the function that maximizes γ with respect to x_i by solving $\frac{\partial \gamma}{\partial x_i}(x_i, y_i) = 0$. This yields

$$\begin{aligned} f(x_i) &= \frac{k_i x_i^* + y_i^*}{2} - \frac{k_i + 1}{2} x_i \ , \\ g(y_i) &= \frac{k_i x_i^* + y_i^*}{2k_i} - \frac{k_i + 1}{2k_i} y_i \ . \end{aligned}$$

Then, for any fixed x_i, the optimal y_i is determined by $y_i = f(x_i)$, and for any fixed y_i, the optimal x_i is determined by $x_i = g(x_i)$. Then, define $\tilde{x}_i$ and $\tilde{y}_i$ as follows:

$$\begin{aligned} \tilde{y}_i &= \operatorname*{argmax}_{y_i \geq 0,\ g(y_i) \geq 0} \gamma(g(y_i), y_i) \\ \tilde{x}_i &= \operatorname*{argmax}_{x_i \geq 0,\ f(x_i) \geq 0} \gamma(x_i, f(x_i)) \ . \end{aligned}$$

We see that $\gamma(g(y_i), y_i)$ and $\gamma(x_i, f(x_i))$ are convex, and $\gamma(g(\tilde{y}_i), \tilde{y}_i) \geq \gamma(\tilde{x}_i, f(\tilde{x}_i))$, where $\tilde{y}_i =$

$\frac{k_i x_i^* + y_i^*}{2}$. Therefore,

$$\max_{x_i \geq 0, y_i \geq 0} \gamma(x_i, y_i) = \gamma(g(\tilde{y}_i), \tilde{y}_i) = \frac{(k_i x_i^* + y_i^*)^2}{4} \ .$$

After applying a similar analysis for roads $\ell < i \leq n$,

$$\begin{aligned}
&\beta(c, z^*) \\
&\leq \frac{1}{4} \frac{\sum_{i=1}^{\ell} \rho_i (k_i x_i^* + y_i^*) + \sum_{i=\ell+1}^{n} \sigma_i (x_i^* + k_i y_i^*)}{\sum_{i=1}^{\ell} \rho_i (x_i^* + y_i^*) + \sum_{i=\ell+1}^{n} \sigma_i (x_i^* + y_i^*)} \\
&= \frac{k}{4} \frac{\sum_{i=1}^{\ell} \rho_i (k_i x_i^* + y_i^*) + \sum_{i=\ell+1}^{n} \sigma_i (x_i^* + k_i y_i^*)}{\sum_{i=1}^{\ell} \rho_i (k x_i^* + k y_i^*) + \sum_{i=\ell+1}^{n} \sigma_i (k x_i^* + k y_i^*)} \\
&\leq \frac{k}{4} \ ,
\end{aligned} \tag{12}$$

where $\rho_i \triangleq a_i(k_i x_i^* + y_i^*)$ and $\sigma_i \triangleq a_i(x_i^* + k_i y_i^*)$. The fact that $\frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} v_i} \leq 1$ when $0 \leq w_i \leq v_i$ implies Equation (12), since $k \geq k_i \geq 1 \ \forall i$. We apply Theorem 1 to find a price of anarchy bound of $\frac{4}{4-k}$ and bicriteria bound of $1 + \frac{k}{4}$. □

5.3 Nonseparable costs

Having discussed pairwise separable costs (Definition 2), where the delay on each road depends only on the vehicles on that road, we now consider nonseparable costs (Definition 3). As an example, consider a series of roads, each one feeding into the next; if one road is fully congested, this will increase the delay on the roads feeding it, resulting in cascading congestion. Another scenario of nonseparable costs is when intersecting streets affect the traffic on each other [22], such as in a signalized intersection that senses traffic and adjusts its duty cycle accordingly. In that case, the volume of traffic on a road will affect the delay on the perpendicular road.

To put this in more concrete terms, consider a road feeding into another narrower road. We model the congestion on the second road as comparatively affecting that on the first road by a factor of μ. This results in a cost function of

$$c(z) = \begin{bmatrix} k_1 a_1 & a_1 & \mu k_2 a_2 & \mu a_2 \\ k_1 a_1 & a_1 & \mu k_2 a_2 & \mu a_2 \\ 0 & 0 & k_2 a_2 & a_2 \\ 0 & 0 & k_2 a_2 & a_2 \end{bmatrix} z + b \ .$$

With this motivation, we consider the affine cost functions $c(x) = Ax + b$, where A is no longer a 2x2 block-diagonal matrix. We consider the case that A can be written as the sum of Q, a (2x2) block diagonal matrix with strictly positive block diagonal entries, and P, a positive definite matrix.[2]

We describe the bounds we can establish under these conditions in the following theorem:

Theorem 3. *Let z^{EQ} be an equilibrium of a nonatomic congestion game with cost function $c(z) = Az + b$. Suppose A can be split into Q, which is a (2x2) block diagonal matrix with strictly positive entries on the block diagonal, and P, which is positive definite, such that $A = Q + P$. Let k be the maximum degree of asymmetry for the cost function defined by Q.*

(a) If z^{OPT} is a social optimum for this game, and if $k < 4$, then $C(z^{EQ}) \leq (\frac{4}{4-k} + \eta^2)C(z^{OPT})$, where $\eta^2 = \lambda_{max}(S^{-1/2}P^T S^{-1} P S^{-1/2})$ and $S = (P + P^T)/2$.

(b) If w^{OPT} is a social optimum for the same game with $2 + \frac{k}{4}$ times as many players of each type, then $C(z^{EQ}) \leq C(w^{OPT})$.

[2] Note that if P is a diagonal dominant mapping, *i.e.* $P_{ii} > \frac{1}{2}\sum_{j \neq i} |P_{ij} + P_{ji}|$, then it is positive definite [33]. In this case, in order to also guarantee that the block diagonal components of Q have strictly positive entries, we require

$$A_{ii} > \frac{1}{2} \sum_{j \neq i, i-1} |A_{ij} + A_{ji}| \text{ for } i \text{ even },$$
$$A_{ii} > \frac{1}{2} \sum_{j \neq i, i+1} |A_{ij} + A_{ji}| \text{ for } i \text{ odd }.$$

This, however, is a sufficient but not necessary condition.

Proof. For part (a), we split the price of anarchy into two components, as

$$\begin{aligned}
\frac{C^{EQ}}{C^{OPT}} &= \sup_{z\in\mathcal{X}} \frac{(Az^{EQ}+b)^T z^{EQ}}{(Az+b)^T z} \\
&= \sup_{z\in\mathcal{X}} \frac{((Q+P)z^{EQ}+b)^T z^{EQ}}{((Q+P)z+b)^T z} \\
&\le \sup_{z\in\mathcal{X}} \frac{(Qz^{EQ}+b)^T z^{EQ}}{(Qz+b)^T z} + \sup_{z\in\mathcal{X}} \frac{(Pz^{EQ})^T z^{EQ}}{(Pz)^T z} && (13) \\
&\le \frac{1}{1-\beta(\mathcal{C}_k)} + \eta^2 && (14) \\
&= \frac{4}{4-k} + \eta^2 \ . && (15)
\end{aligned}$$

Inequality (13) follows from all latencies being nonnegative, (14) follows from [27] and [26] (see the comment on page 2 about the price of anarchy for costs with no constant term), and the (15) is proved in the proof of Theorem 2.

For part (b), we use the same notion of $\beta(\mathcal{C})$ as in the proofs for Theorems 1 and 2, as follows:

$$\begin{aligned}
\beta(c,v) &= \max_{z\in\mathbb{R}^{2n}_{\ge 0}} \frac{\langle c(v)-c(z), z\rangle}{\langle c(v), v\rangle} \\
&= \frac{\max_{z\in\mathbb{R}^{2n}_{\ge 0}} \langle (Q+P)(v-z), z\rangle}{\langle (Q+P)v+b, v\rangle} \\
&\le \frac{\max_{z\in\mathbb{R}^{2n}_{\ge 0}} \langle Q(v-z), z\rangle}{\langle (Q+P)v+b, v\rangle} + \frac{\max_{z\in\mathbb{R}^{2n}_{\ge 0}} \langle P(v-z), z\rangle}{\langle (Q+P)v+b, v\rangle} \\
&\le \frac{\max_{x\in\mathbb{R}^{2n}_{\ge 0}} \langle Q(v-z), z\rangle}{\langle Qv+b, v\rangle} + \frac{\max_{z\in\mathbb{R}^{2n}_{\ge 0}} \langle P(v-z), z\rangle}{\langle Pv+b, v\rangle} \\
&= \beta(c_1,v) + \beta(c_2,v)
\end{aligned}$$

Here c_1 and c_2 represent cost functions drawn from $\mathcal{C}_k$ and $\tilde{\mathcal{C}}$, respectively, where k is the maximum degree of asymmetry of the cost function $c(z) = Qz + b$ and $\tilde{\mathcal{C}}$ denotes the set of monotone cost functions.

De Palma and Nesterov [33] show that a cost function $c(z)$ is monotone if $c'(z)$ is positive definite. Furthermore, Correa *et. al.* show that a class $\mathcal{C}$ consisting of monotone cost functions has

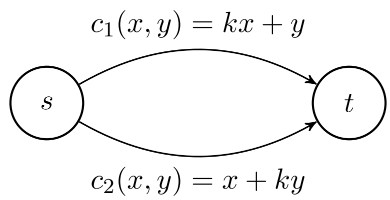

Figure 3: Example of a road network with two-sided asymmetry.

$\beta(\mathcal{C}) \leq 1$. This is easily demonstrated as follows. Using (1) with $z, v \in \mathbb{R}^{2n}_{\geq 0}$,

$$1 \geq \frac{\langle c(v) - c(z), z\rangle}{\langle c(v) - c(z), v\rangle} \geq \frac{\langle c(v) - c(z), z\rangle}{\langle c(v), v\rangle} \geq \beta(c, v) \ .$$

Because of this,

$$\begin{aligned}
\beta(\mathcal{C}) &= \sup_{c \in \mathcal{C}, v \in \mathcal{X}} \beta(c, v) \\
&\leq \sup_{c \in \mathcal{C}_k, v \in \mathcal{X}} \beta(c, v) + \sup_{c \in \tilde{\mathcal{C}}, v \in \mathcal{X}} \beta(c, v) \\
&\leq \frac{k}{4} + 1 \ .
\end{aligned}$$

Here $\tilde{\mathcal{C}}$ denotes monotone cost functions. Applying Theorem 1 completes the proof. □

6 TIGHTNESS OF THE BOUND

In this section, we discuss the tightness of the bound derived in Section 5.2. In Section 6.1 we provide two examples: Example 2 shows that our price of anarchy is tight for $k = 2$ and our bicriteria bound is tight for $k = 4$ when there can be two-sided asymmetry, *i.e.* k_i can be greater or less than 1. In a more realistic scenario, we expect autonomous vehicles to result in the same amount or less congestion than regular cars for all roads. In light of this, we provide Example 2 of *one-sided* asymmetry, in which $k_i \geq 1 \ \forall i$. In Section 6.2, we discuss the tightness of the bound with respect to both of these scenarios.

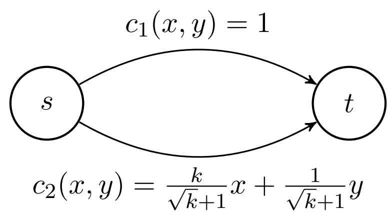

Figure 4: Example of a road network with one-sided asymmetry.

6.1 Examples

Example 2. *Consider the traffic network in Fig. 3, which is parameterized by the degree of asymmetry, k. We wish to transport 1 unit regular traffic and 1 unit smart traffic across the network.*

The worst-case Nash equilibrium has all regular traffic on the top link and all the smart traffic on the bottom link, for a cost of $C^{EQ} = 2k$. The optimal routing has this routing reversed, for a cost of $C^{OPT} = 2$. This gives us $\frac{C^{EQ}}{C^{OPT}} = k$.

We find the bicriteria by finding how much traffic we could optimally route for a cost of $2k$. Consider p units regular and p units of smart vehicles, which would have optimal routing cost $2p^2$. Setting $2p^2 = 2k$, we find the bicriteria is $\sqrt{k}$.

Example 3. *Consider the traffic network in Fig. 4, which is parameterized by k. Here we wish to transport $\frac{1}{\sqrt{k}}$ units regular traffic and 1 unit smart traffic across the network.*

At the Wardrop Equilibrium, all traffic will take the bottom route for a delay of 1, which gives us cost $C^{EQ} = \frac{1}{\sqrt{k}} + 1$. In optimal routing we have regular traffic on top and smart traffic on the bottom. This gives us $C^{OPT} = \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k}+1}$, giving us a PoA of $1 + \frac{k}{2\sqrt{k}+1}$.

We find the bicriteria by setting the cost of routing p times as much traffic optimally equal to the original cost at equilibrium. This gives us $p = \frac{(-1+\sqrt{1+4\sqrt{k}})(1+\sqrt{k})}{2\sqrt{k}}$.

6.2 Discussion

We begin by discussing the price of anarchy. Our bound for price of anarchy is $\frac{4}{4-k}$, and example 2 shows a price of anarchy of k and example 3 shows a price of anarchy of $1 + \frac{k}{2\sqrt{k}+1}$. For the bicriteria, our bound is $1 + \frac{k}{4}$. Example 1 provides a bicriteria of $\sqrt{k}$ and example 2 has a bicriteria

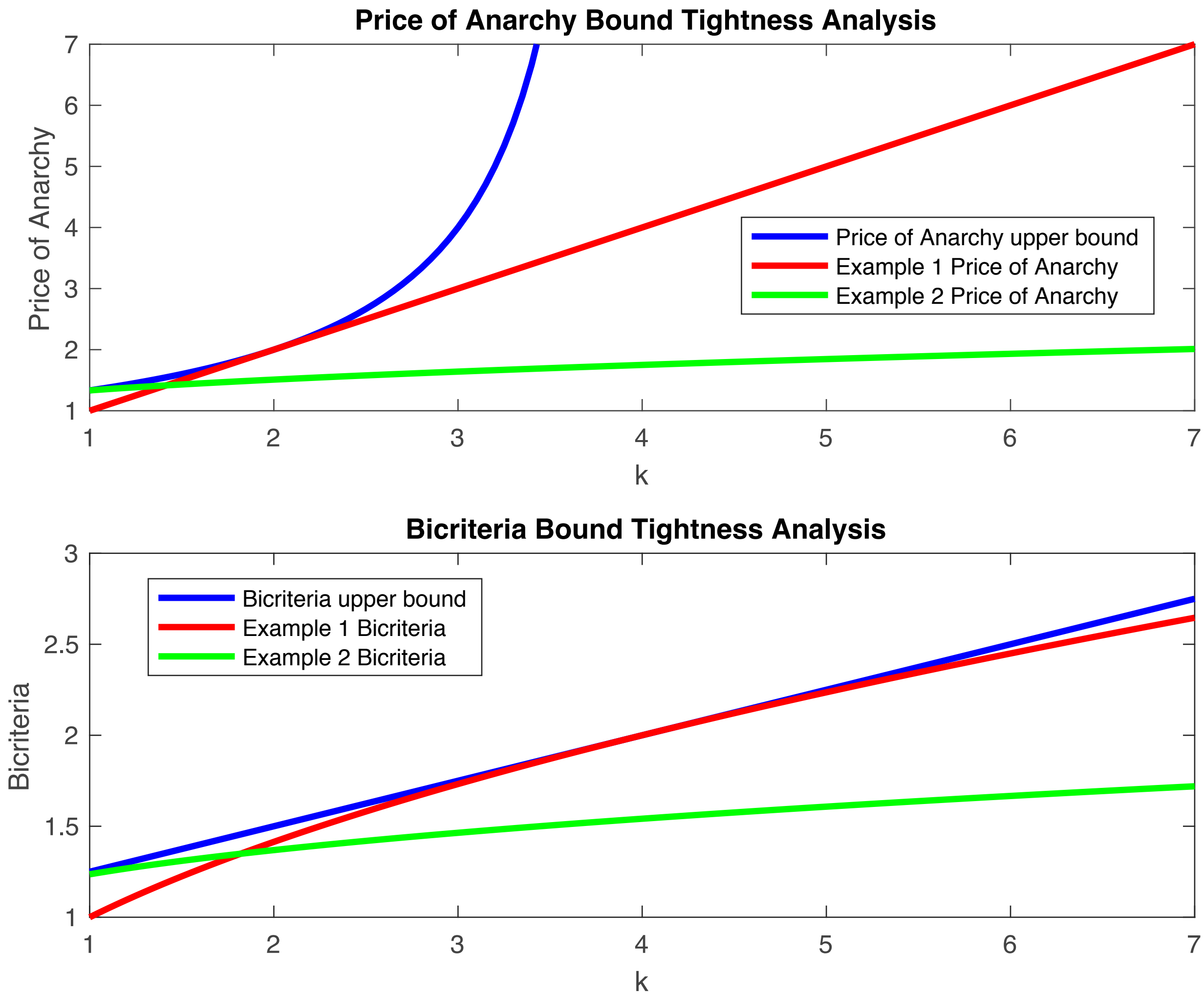

Figure 5: Tightness of the bounds for price of anarchy and bicriteria. The PoA bound is tight for $k = 1, 2$ and the bicriteria bound is tight for $k = 4$.

that scales with $k^{1/4}$.

When $k = 1$, price of anarchy bound recovers the classical bound found in [20]. Further, the examples show that the price of anarchy bound is tight for $k = 2$ and the bicriteria bound is tight for $k = 4$.

Figure 5 illustrates these comparisons. In both cases, our upper bound diverges from these lower bounding examples for large k. Therefore, it is unknown if our bound is tight in that regime. However, realistic circumstances lead to $k \approx 2.5$, which is in the near-tight region for both price of anarchy and bicriteria.

It is worth noting that under the construction in [27] and in Theorem 1, there can be no bound on the price of anarchy for networks with $k \geq 4$. Observe that in Example 2 for $k = 4$, the bicriteria is 2. This means that $\beta(\mathcal{C}_{k=4}) \geq 1$, so the bound on the price of anarchy does not hold.

7 NUMERICAL STUDIES

7.1 Revisiting Capacity Models

To have a more accurate capacity model, we first revisit the results derived in Equations (2) and (3). In capacity model 1, an autonomous vehicle is indiscriminate in reducing its headway when following other vehicles. Let m be the capacity of the road when fully utilized by regular vehicles, and M be the capacity of the road when fully utilized by autonomous vehicles. Let α be the average proportion of smart vehicles on the road. As mentioned before, the capacity of the road under autonomy level α is approximated by

$$C(\alpha) = \frac{1}{\alpha M^{-1} + (1-\alpha) m^{-1}}. \tag{16}$$

Note that, for a single lane road of length d, where regular vehicles assume headway h_r and autonomous vehicles assume headway h_s, we have $M = d/h_s$ and $m = d/h_r$. Let the length of every vehicle be l. We can then express a more accurate capacity model as

$$C(\alpha) = \frac{d}{\alpha h_s + (1-\alpha)h_r + l}. \tag{17}$$

The physical significance of h_r and h_s can be appreciated below in Fig. 6.

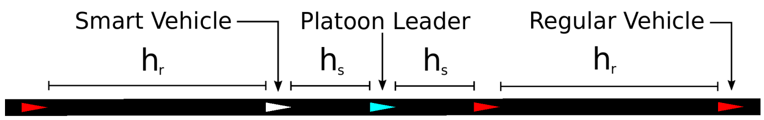

Figure 6: A representation of car interactions in SUMO according to capacity model 1. In this model, an autonomous vehicle will always follow with a distance h_s to the car in front of it. A regular vehicle will follow with headway h_r.

In capacity model 2, an autonomous vehicle adjusts its headway according to the technology of the car it is following; it only reduces its headway when following another autonomous vehicle. Using the same notation as above, we find that

$$C(\alpha) = \frac{1}{\alpha^2 M^{-1} + (1-\alpha^2)m^{-1}} = \frac{d}{\alpha^2 h_r + (1-\alpha^2)h_s + l}. \tag{18}$$

The physical significance of h_r and h_s can be appreciated in Fig. 7 below.

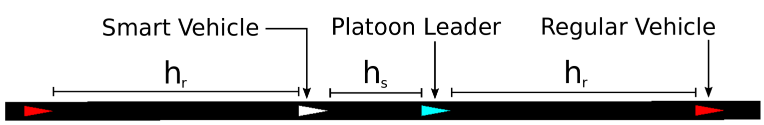

Figure 7: A representation of car interactions in SUMO according to capacity model 2. In this model, an autonomous vehicle will only follow with a distance h_s to the car in front of it if that car is also autonomous. A regular vehicle will always be followed with distance h_r.

7.2 Validation of Models In SUMO

In this subsection, we discuss the SUMO validation and configurations.[3]

The most basic road network was used to validate the capacity models - a single-lane road in a straight line. Below, we outline the basic configuration files needed to define this scenario.

1. `/single_road/network/single.net.xml`
    - This file defines the road graph, including the locations of vertices in a plane, the edges between vertices, and the speed limit along those edges.
2. `/single_road/network/single.rou.xml`
    - Types of vehicles are defined here according to a car following model, color, acceleration parameters, impatience, length of vehicle, maximum speed, minimum gap from vehicle immediately in front, and headway.
    - Instances of these vehicles on the road are also defined here. This can be done in multiple ways, but for this study it was useful to define a "flow," where a proportion of different vehicle types is specified and the traffic is generated from this distribution.
3. `/single_road/network/single.det.xml`
    - This file defines sensors on the road, including the locations of the sensors on specific edges of the road graph, the frequency of detection, and the output files. Note that the output files weren't necessary, as an interface from Python was available to directly talk to these sensors.

[3] All source files can be found in https://github.com/davidrower/collaboration_pedarsani.

An example of a sensor in the graphical rendering of SUMO is presented in Fig. 8.

Figure 8: A sensor as displayed in the GUI of SUMO. The yellow box is the sensor, which can be listened to directly through a Python script. The red triangle represents a car.

Overview of TraCI. In order to gather data quickly and effectively from SUMO, TraCI, a "Traffic Control Interface," was released. TraCI allows you to directly observe and manipulate instances of SUMO via several supported programming languages, including Python. This was used to automate the running of several instances of SUMO, and to collect data from each of those instances.

Our Scenario. A single-lane road in a straight line was defined, and two sensors were defined along the road. One sensor was placed roughly 1/5 of the way down the road, the other was placed near the end of the road (but not at the end, as this led to a bug). The first sensor was placed such that the traffic flow from the source node would equilibrate before reaching the sensor. The number of cars which passed each sensor could be counted, and the number of cars on the patch of road between the two sensors could be calculated from the difference between the sensor counts.

Methodology. This number of cars on the patch between the sensors was recorded several times over the duration of a simulation, and the average and standard deviation were computed. The average number of cars on the road was plotted against the road capacity for a sampling of autonomy levels for two scenarios: parameterizations matching the descriptions of capacity models 1 and 2. These results are plotted in Fig. 9 and Fig. 10.

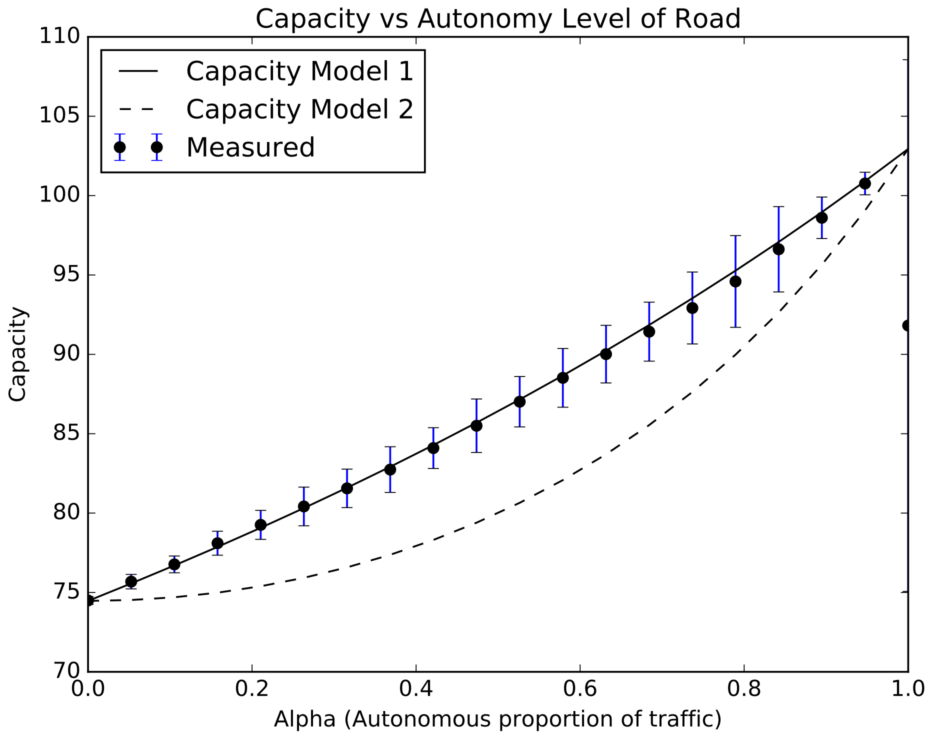

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Figure 9: Road capacity as a function of autonomy level with parameterization of capacity model 1. In this model, autonomous vehicles are indiscriminate in reducing their headway. There is a very strong agreement between the model and the measurements.

Discussion. There are two common features in these studies. The standard deviation of the road capacity is a maximum when the traffic is very mixed. It takes on low values when the traffic tends to be very regular or very autonomous. The second common feature is an artifact of the finite length of the road and only affects the simulations for autonomy level $\alpha = 1$.

As vehicles are added to the road, they need to accelerate to catch up to the (infinite) platoon in front of them. Over time, the gap between a newly injected vehicle and the vehicle injected before it grew too large to be overcome by the acceleration of the vehicles. This problem wasn't

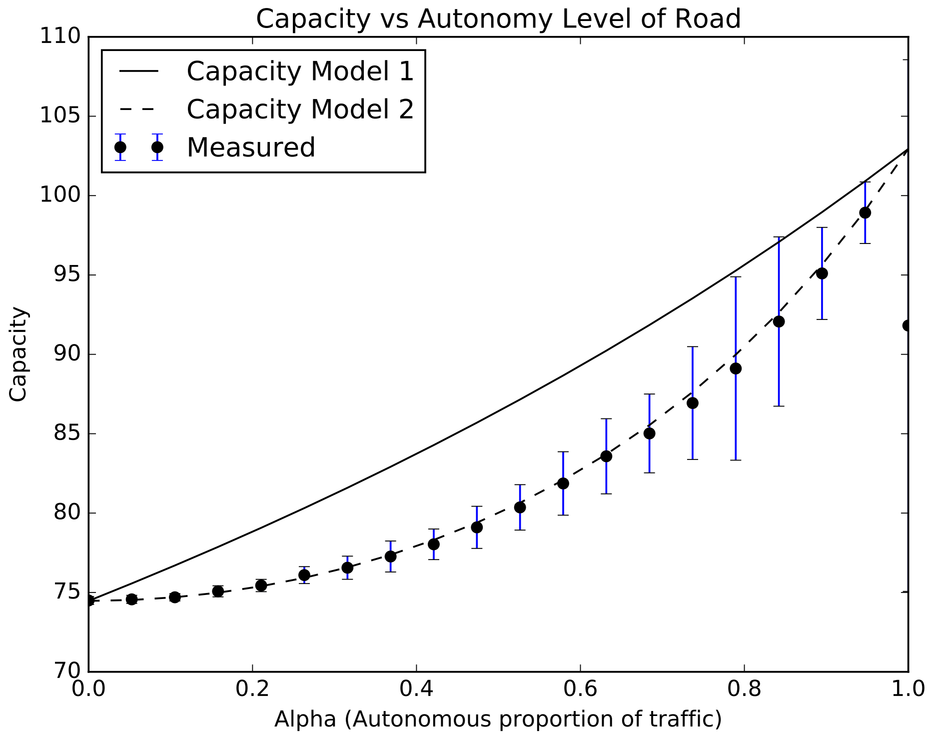

Figure 10: Road capacity as a function of autonomy level with parameterization of capacity model 2. In this model, autonomous vehicles only reduce their headway if the leading car is also autonomous. There is a very strong agreement between the model and the measurements.

present in simulations with lower values of α since regular vehicles would be injected often enough to keep this gap from growing too large.

8 CONCLUSIONS

In this report, we developed capacity models for transportation networks with mixed autonomy. Using these models, we presented pairwise separable and nonseparable cost functions for traffic networks under mixed autonomy. We demonstrated that the price of anarchy and bicriteria is unbounded without constraints on the asymmetry in the difference in how the additon of smart and regular vehicles affects congestion. We then established bounds for the price of anarchy and bicriteria, parameterized by the degree of asymmetry of the network, for both the case of pairwise separable and nonseparable costs, under certain conditions. We analyzed the tightness of the bounds for the pairwise separable case and demonstrate that they are tight for certain degrees of asymmetry of the network. Finally, we presented simulations results via SUMO that showed tight agreement between our theoretical models and practice.

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