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16. ABSTRACT

There has recently been considerable interest in cruising for curbside parking as a major contributor to traffic congestion in the downtown areas of major cities. This paper focuses on the expected search time for a curbside parking space. The literature has employed three different approaches to estimate expected cruising-for-parking time: direct measurement, inference based on the equilibrium condition that (for the marginal parker) the expected cost of curbside parking equals the expected cost of garage parking, and inference based on the observed occupancy rate of curbside parking and an assumed statistical relationship between expected cruising-for-parking time and the curbside parking occupancy rate. The last approach typically obtains estimates of expected cruising-for-parking times that are lower, and with high occupancy rates much lower, than those estimated using the other two approaches. This paper takes a step towards resolving this inconsistency by demonstrating, through computer simulation of cars cruising for parking around a circle in stochastic steady state, that an approximating assumption in the derived statistical relationship between expected cruising-for-parking time and the curbside parking occupancy rate leads to underestimation of average cruising-for-parking time, and at high occupancy rates very considerable underestimation. The paper also identifies several "effects" that contribute to the approximating assumption being an increasingly poor one as the occupancy rate increases.

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**Towards Inferring Welfare Changes from
Changes in Curbside Parking Occupancy Rates**

A Theoretical Analysis Motivated by
SFpark and LA Express Park

Final Report

Richard Arnott

July 4, 2016

The Terms of the Task Order

The Department of Transportation (Caltrans) Task Order was based on the original UCTC research grant proposal, which was written in April 2013. The Task Order was to develop a conceptual and analytical framework for calculating the optimal curbside parking occupancy rate, and to illustrate its application to a spatially uniform downtown area in stationary state. The performance tests were to be three technical reports. The first was to present the conceptual/analytical framework for calculating the optimal target curbside parking occupancy rate in a spatially uniform downtown area, treating alternative search strategies. The second was to apply an agent-based traffic microsimulation model to calculate numerically the optimal curbside parking occupancy rate under alternative scenarios, including alternative search strategies. The third was to apply queuing theory to derive analytical solutions of the optimal curbside occupancy rate under alternative scenarios, including alternative search scenarios.

Actual Output Delivered

The actual output under the Task Order was one published paper and one technical report.

- The published paper was

Arnott, R. 2014. On the optimal target curbside parking occupancy rate. *Economics of Transportation* 3: 133-144.

The *Economics of Transportation* is the leading international journal in the economics of transportation. The paper developed a conceptual and analytical framework for calculating the optimal curbside parking occupancy rate around a circle.

- The technical report was

"Cruising for Parking Around a Circle", Richard Arnott and Parker Williams.

The technical report employed stochastic microsimulation to investigate the probability distribution of cruising-for-parking times around a circle, as a function of the expected occupancy rate, the probability distribution of parking duration, and the number of parking space.

Electronic versions of both the journal article¹ and the technical report are included as attachments to the final report e-mail.

Informal Description of the Research Findings and Their Importance

1. "On the optimal target curbside occupancy rate"

Consider a uniform downtown area in which the demand function for curbside parking is uniform over time. What is the optimal curbside meter rate? If a local transportation

¹ Please note that funding from the grant is not acknowledged since the research was undertaken during a period when the grant funds were frozen due to contract negotiations between Caltrans and the University of California, Berkeley.

authority sets the curbside meter rate too low, there will be excessive cruising for parking. Not only does this result in wasted time cruising for parking (and walking from the parking space to the destination, and back) but also cars cruising for parking contribute to traffic congestion. If a local transportation authority sets the curbside rate too high, trips that have social value will not be undertaken.

Donald Shoup proposed that the local transportation authority adjust the meter rate until an optimal target curbside parking occupancy rate is achieved. What is the optimal curbside parking occupancy rate? The published paper is the first to develop the conceptual and analytical framework to answer this question. The short answer is that there is not a magic number for the optimal target curbside occupancy rate. Rather, the optimal rate increases with the demand for curbside parking relative to capacity. Imagine plotting marginal social benefit and marginal social cost against the occupancy rate. The optimal occupancy rate occurs at their intersection point. Since the marginal social benefit curve is downward sloping and the marginal social cost curve is upward sloping, an increase in demand for curbside parking, which entails the marginal social benefit curve shifting up, results in an increase in the optimal occupancy rate.

The San Francisco curbside parking experiment, SFpark, adaptively adjusted the curbside meter rate on a block-by-block and time-of-day basis until what was judged to be optimal rates were achieved. SFpark found that the mean occupancy rates associated with the optimal meter rates varied considerably, which is consistent with the theoretical results of the grant research. Hopefully SFpark's findings, buttressed by the theoretical results of the grant research, will persuade other jurisdictions that adjusting curbside meter rates to achieve a common target curbside parking occupancy rate is ill advised.

2. *"Cruising for parking around a circle"*

There has recently been considerable interest in cruising for curbside parking as a contributor to downtown traffic congestion. How important is it? This has proved a difficult question to answer since cars cruising for parking cannot be distinguished from the general traffic flow through direct observation. A common approach has been to assume that the probability of finding a particular curbside parking space vacant equals the average curbside vacancy rate. The report refers to this assumption as the "binomial approximation". The report uses stochastic microsimulation to gauge the validity of the approximation. In particular, it compares the probability distribution of cruising-for-parking times that are generated through microsimulation with that implied by the binomial approximation. While this is not the first research to incorporate cruising for parking into a microsimulation model, it is the first to use microsimulation to focus on cruising for parking and to estimate the distribution of cruising-for-parking times, albeit in a very simple setting.

The research has two main findings. The first is that the binomial approximation leads to underestimation of average cruising-for-parking time, and at high occupancy rates to very considerable underestimation. This finding is consistent with the intuition that motivated the research, but the degree of underestimation was unexpected. When the mean

occupancy rate is $2/3$, the actual mean cruising-for-parking time is 140% that obtained under the binomial approximation; the corresponding number with a mean occupancy rate of $5/6$ is 300%, and with a mean occupancy rate of $11/12$ is 850%. These results have two important implications for downtown traffic management. The first is that the time lost due to cruising for parking is indeed potentially considerable. The second is that mean cruising-for-parking time rises considerably more rapidly with the occupancy rate than according to the binomial approximation, being 27 times as high with a mean occupancy rate of $11/12$ as with a mean occupancy rate of $2/3$. The clear implication is that meter rates should be increased on those blocks with high mean occupancy rates in order to avoid high cruising-for-parking costs.

The second main finding was unexpected. Even though each simulation ran 1,000,000 cars through the system, simulating 500 hours of traffic flow, the mean cruising-for-parking time differed significantly between simulations with the same parameter values. This result led to the hypothesis, which was confirmed by more detailed examination, that mean cruising-for-parking time is strongly influenced by rare but extreme events ("disasters") in which curbside parking becomes almost gridlocked. Purely by statistical accident, there are times when not only does curbside parking become saturated but also a stock of cars cruising for parking accumulates. To some extent the finding is an artifact of the design of the simulations, which did not provide drivers with the option of parking off street when curbside parking is very difficult to find. Nevertheless, the finding does underscore the importance of aggregate stochasticity in curbside parking, which signals the potential value of the truly responsive curbside parking pricing advocated by William Vickrey in the 1950's: Adjust curbside parking fees on a real-time basis in response to realized local occupancy rates, so as to allow for one vacant parking space on each block almost always. Doing so would nip incipient curbside parking gridlock in the bud.

Comparison of Actual Output Delivered to That Specified in the Task Order

In terms of topic, the actual output delivered corresponds closely to that specified in the task order. Unfortunately, not as much progress was made as was hoped for. The most significant task order goal that was not met was that neither the research on the optimal target curbside parking occupancy rate nor that on cruising for parking was extended to two-dimensional space. The PI underestimated the conceptual difficulty in doing so. Cars that cruise for parking in one direction on the outside of a circle have no option but to continue cruising. However, cars that cruise for parking in an isotropic downtown area have many options. The simplest is to drive to the destination block and then to circle the block. At low expected occupancy rates, that is likely the best option, but at higher expected occupancy rates it is almost certainly not. Solving for the optimal cruising-for-parking search strategy appears intractable. The simplest approach is to ascertain through simulation which of a number of heuristic strategies, when used by everyone, works best at different mean occupancy rates. A more sophisticated approach is to investigate the evolutionarily stable strategy, again by simulation. Cellular automata are endowed with different search strategies. Those that search most effectively reproduce most rapidly, while those that search least effectively die off. Another more sophisticated approach is

allow drivers to experiment and learn. A common difficulty with these more sophisticated approaches is that the effectiveness of a strategy for one agent may depend on the mix of strategies employed by other agents. Thus, the equilibrium or the optimum, as the case may be, may entail a mix of strategies. "Cruising for parking around a circle" uncovered the considerable stochastic variability of cruising-for-parking times, and the importance at high mean occupancy rates of extreme events. Intuitively, adjustment may be unstable if agents' choices of strategy are sensitive to such events. In principle, an alternative is to observe the cruising-for-parking strategies that drivers actually use. The difficulty with this approach is that cruising-for-parking strategies are difficult to observe simply because it is difficult to observe when a car is cruising for parking.

Another task order goal that was not met was that closed-form analytical solution of the distribution of cruising-for-parking times was not obtained, even for one dimension. It turned out that neither queuing theory nor Markov process theory has yet been developed to the point where analytical solutions can be obtained without making approximating assumptions that might alter qualitatively the solution. The PI decided instead to devote more time to stochastic microsimulation, because it was bearing fruit.



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journal homepage: www.elsevier.com/locate/ecotraOn the optimal target curbside parking occupancy rate[Ⓓ]

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a b s t r a c t

Donald Shoup, following Vickrey, has long advocated cashing out free and underpriced parking. How should this be implemented for curbside parking in practice, considering the stochasticity of curbside parking vacancies? Shoup has proposed adjusting meter rates such that, for each block and time period, a target (average) curbside parking occupancy rate of 85% is achieved. This paper develops a simple structural model of stochastic steady-state curbside parking in an isotropic space, solving for the surplus-maximizing occupancy rate and the corresponding meter rate. By increasing curbside occupancy, a curbside parker imposes a curbside parking externality. The optimal meter rate internalizes this externality. The central comparative static result is that, *ceteris paribus*, the optimal occupancy and meter rates are higher, the higher is demand relative to curbside parking capacity. This suggests that, in practice, the occupancy rate should be higher in more trafficked locations and at busier times of the day.

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1. Introduction

Even on a particular city block at a particular time of day, the number of vacant curbside parking spaces varies substantially from day to day. At a point in time, the block's curbside parking vacancy rate is the realization of complex curbside parking arrival (birth) and departure (death) stochastic processes. There may be a special event on that block that generates a higher than usual arrival rate for a period of time before that event starts, and a concentration of departures when it is over. Or purely by chance, an abnormally high number of people may choose to have lunch at a particular restaurant on that block. To further complicate matters, there is spatial autocorrelation in the vacancy rate on neighboring blocks, as drivers, unable to find curbside parking on their destination block, search/cruise for parking on neighboring blocks.

The stochasticity of vacant curbside parking spaces is practically important. Even when the average vacancy rate on a particular city block at a particular time of day is, say, 10%, there will be days when drivers whose destination is on that block have to spend considerable time cruising for a curbside parking space. So as to avoid being late for an appointment, a driver may respond

to this lack of reliability¹ in curbside parking search time by departing home earlier and/or starting to search for parking well before reaching the destination block.

Vickrey (1954) was the first economist to address the importance of the stochastic nature of the curbside parking vacancy rate. He advocated responsive curbside parking pricing to deal with the phenomenon. The parking meters on a block would be simultaneously monitored, and the meter rate would be adjusted responsively to achieve an *ex post* curbside vacancy rate, such that there would almost always be a vacant curbside parking space on each block, which would virtually eliminate the time wasted in cruising for parking and the added congestion it causes. Vickrey's proposal never went beyond the drawing board because it was technologically ahead of its time. Recently Shoup (Shoup 1999, 2006; King et al., 2007) has been advocating differentiating the curbside meter rate by block and time of day so as to achieve a common target curbside parking occupancy rate (hereafter, target occupancy rate); he has proposed a rate of 85%. Implementing his proposed scheme would require extensive data collection but no high technology. A modified version of his proposed scheme is being implemented on an experimental basis in San Francisco (SFpark.org; Millard-Ball et al., 2014; Pierce and Shoup, 2013).

^{*}This paper is dedicated to the memory of Herbert Mohring, in recognition of his seminal contributions to urban transport economic theory.

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¹Since this paper employs social surplus analysis, it implicitly assumes that drivers are risk neutral. Even though drivers are risk neutral, there are still costs associated with the stochasticity, and hence unreliability, of finding a curbside parking space. Taking into account that most drivers are risk averse would add another cost of unreliability.

Nomenclature	
$a\delta x; e; z$	arrival time at destination
e	departure time relative to beginning of scheduled appointment
f	meter rate
$g\delta z; Q$	pdf of cruising-for-parking distance, conditional on Q
r	throughput
s	cruising-for-parking speed
uc	user cost
v	in-transit travel speed
w	walking speed
x	distance before destination cruising for parking initiated
z	cruising for parking distance (random variable)
$D\delta F$	trip demand per unit time-distance
$E_1; E_2$	equilibria
$E\delta x; Q$	expected in-transit travel time
$ES\delta Q$	expected cruising-for-parking time
$ESDC\delta x; e; Q$	expected schedule delay cost
$ETTC\delta x; Q$	expected travel time cost
$EW\delta x; Q$	expected walking time cost
F	full price of a trip
L	visit duration
$M\delta x; e$	cruising-for-parking distance corresponding to on-time arrival
MSC	marginal social cost
P	parking spaces per unit distance (curbside parking capacity)
Q	curbside parking occupancy rate
TC	total cost
$X\delta p$	social benefit as a function of throughput
α	value of travel time
β	value of time early
γ	value of time late
δ	distance from origin to destination
η	demand shifter
λ	Lagrange multiplier on steady-state condition
Z	intermediate notation

Averaged over time and space, does a target occupancy rate of 85% maximize social surplus, or should it be higher or lower? Should the target occupancy rate vary depending on the time of day, perhaps being lower in the morning when the expected arrival rate exceeds the expected departure rate, or depending on location, perhaps being higher at locations with shorter average parking durations?

This paper takes a first step towards determining the optimal (surplus-maximizing) target occupancy rate analytically. Assume that space is isotropic and that the economy is in stochastic steady state, so that the optimal target occupancy rate is invariant over time and space. Its determination can be viewed as the solution of a problem involving three modules. The first relates to the outcomes of alternative search strategies, taking as given the probability distribution of different patterns of parking occupancy over space. The second derives the probability distribution of different patterns of occupancy over space from the stochastic processes determining trip generation and termination, under alternative search strategies. The third derives the surplus-maximizing target occupancy rate under alternative search strategies. Solution of this problem would be formidably difficult. To generate a problem that is manageable, I construct a model, building on that in [Arnott and Rowse \(1999\)](#), which assumes that parking spaces are uniformly distributed around the circumference of a circle, trip origins are generated by a time-invariant Poisson process at a rate that is uniform around the circle, the distance between trip origins and destinations is constant, cars travel in only one direction towards the destination and in the opposite direction on the return journey, and the visit length at the destination is constant. Under these assumptions, a driver's search strategy is simple: after initiating search, take the first vacant parking space encountered. A driver then has only two decisions to make, how long before her appointment time to initiate her trip, and how far before her destination to initiate search. The probabilities of encountering the first vacant space at the first parking space, the second parking space, and so on, after parking search is initiated, as a function of the average occupancy rate (averaged over time and space), can then be solved for, at least computationally.

The parking planner controls the curbside occupancy rate only indirectly via the curbside meter rate. Adding a demand function relating the Poisson rate at which trips are initiated to the expected full price of a trip permits determination of equilibrium

and social surplus, as functions of the curbside meter rate, and thence of the optimal curbside meter rate and the optimal target occupancy rate.

Because the underlying stochastic processes generating the actual patterns of curbside parking occupancy vary over both space and time and are much more complex than the simple stochastic process assumed in the model, and because actual two-dimensional parking search strategies are much more complex than the simple one-dimensional search strategy implied by our model, considerable work – data collection and analysis, and theoretical development – will need to be done before optimal target occupancy rates can be determined in policy practice. I hope nonetheless that this paper adds value in providing some conceptual foundation for their determination.

[Section 2](#) provides a brief review of relevant literature. [Section 3](#) presents the model. [Section 4](#) derives analytically the optimal target curbside occupancy rate, taking as given the probability distribution of the number of occupied curbside parking spaces searched prior to finding a vacant space, conditional on the average occupancy rate. [Section 5](#) discusses directions for future research, and [Section 6](#) concludes.

2. Literature review

Several papers have investigated models of rush-hour traffic dynamics in which individuals have a common desired arrival time at a common destination, are perfectly informed about the availability of curbside parking spaces on a radial artery, and can choose between vacant parking spaces. Arriving earlier provides a larger choice set of vacant parking spaces but increases schedule delay. [Arnott et al. \(1991\)](#) consider such a model of morning rush-hour travel to a common downtown location, with bottleneck congestion upstream of downtown parking spaces. By varying the meter rate across time and location, the planner can control the order in which parking spaces are occupied, which affects the time pattern of congestion at the bottleneck. [Zhang et al. \(2008, 2011\)](#) and [Qian et al. \(2012\)](#) provide various extensions of [Arnott et al. \(1991\)](#) to examine alternative downtown parking policies. [Anderson and de Palma \(2004\)](#) explore a model similar to [Arnott et al. \(1991\)](#) without a bottleneck but with congestible parking on side streets, one section of which considers cruising for parking.

Arnott and Inci (2006, 2010), Arnott and Rowse (2009, 2013), and Arnott et al. (2013) present a series of related models that investigate the interaction between cruising for underpriced curbside parking and traffic congestion in an isotropic downtown area in stationary state. Curbside parking spaces reduce the road space available for travel, and cars cruising for parking contribute to traffic congestion. Arnott and Inci (2006) explore a model with only curbside parking, and Arnott and Inci (2010) examine the stability of the model's equilibria. Arnott and Rowse (2009) consider a model with both curbside and garage parking, and consider spatial competition between parking garages, and Arnott and Rowse (2013) extend that model to consider curbside parking time limits and heterogeneity among parkers. Arnott et al. (2013) present an integrative diagrammatic analysis focusing on first- and second-best optimal curbside parking capacity. All the above Arnott, Inci, and/or Rowse papers assume that expected cruising for parking search time is given by CL/P where C is the density per unit area of cars cruising for parking, L is the parking duration, and P is the density of curbside parking spaces, and that walking time between the parking space and the destination is zero. This specification assumes that, at all locations, parking spaces become available to a driver who is cruising for parking according to a spatially uniform and time-invariant Poisson process with rate $P \rightarrow \delta CL \rho$. Under this assumption, the optimal search strategy is to drive to the destination block and wait until a space becomes available. Since expected cruising for parking time is CL/P , independent of parking location, the optimal search strategy minimizes expected walking distance.

Arnott and Rowse (1999) are the first economics paper to investigate curbside parking search at a microscopic level. Individuals and curbside parking spaces are uniformly distributed around the circumference of a circle. An individual waits at home for offers, each of which provides a fixed award, which can be collected by going to a specified, stochastically determined location and remaining there for a fixed period of time. Upon receipt of an offer, she must immediately decide whether to take it up. Conditional on taking it up, she departs immediately, decides how far from her destination to start cruising for parking, and then takes the first vacant parking space, walking from there to her destination. Having collected the award, she returns to her parking space, drives home, and awaits the next offer. The paper solved for equilibrium curbside parking occupancy rates, demonstrating possible multiplicity of equilibria. A weakness of the paper, which the authors recognized, is that, to achieve tractability it assumes that the probability that a particular curbside parking space is vacant equals the average curbside parking vacancy rate; that is, the authors assumed away the spatial autocorrelation of occupied curbside parking spaces.

The importance of the spatial autocorrelation of occupied curbside parking spaces is the focus of Levy et al. (2013). They compare the results of an analytical model of parking search similar to Arnott and Rowse (1999), PARKANALYST, in which every driver is confronted with averaged conditions, with the results of a traffic microsimulation model, PARKAGENT, which treats the spatial autocorrelation of occupied curbside parking spaces. Their simulation results demonstrate the quantitative importance of taking into account this spatial autocorrelation. Fig. 1 reproduces Fig. 4 of their paper. It plots average cruising for parking time against the average occupancy rate. In PARKANALYST, parking search time becomes significant only when the average occupancy rate is close to 100% (for example, with a occupancy rate of 99%, a driver cruising for parking would expect to drive by 99 occupied parking spaces before locating a vacant space, while with a parking occupancy rate of 90%, a driver cruising for parking would expect to drive by only 9 occupied parking spaces). In PARKAGENT, in contrast, parking search time starts to be non-negligible with an

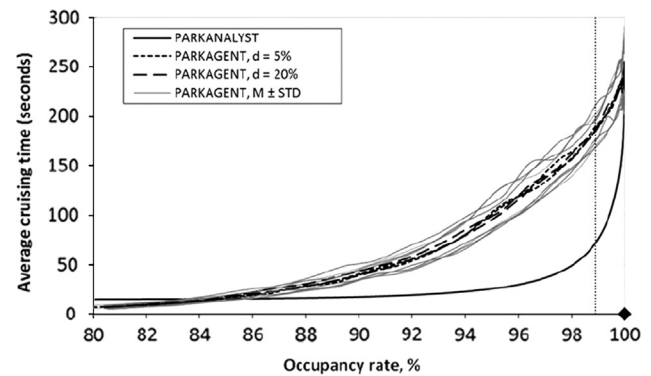


Fig. 1. Average cruising time as a function of the parking occupancy rate. PARKANALYST vs PARKAGENT. This figure is reproduced from Fig. 4 of Levy et al. (2012). In the current context, the difference between the various PARKAGENT graphs is unimportant.

average occupancy rate of about 85%, and at around 93% is approximately the same as that in PARKANALYST with a 99% rate.

3. The model

Consider an isotropic spatial economy organized on the circumference of a circle (an “racetrack economy”) of infinite radius.² Trips are originated around the circle at a uniform Poisson rate that is determined endogenously. Each trip entails travel in one direction³ around the circle to a destination a distance δ from where the trip originated, a visit at the destination of duration L at an appointed time, followed by a return journey in the opposite direction to the trip origin. All trips are by car, and a driver must park her car curbside in the vicinity of her destination, and walk from her parking location to her destination, and later back again before driving back to the trip origin. Curbside parking spaces are uniformly distributed around the circle with density P per unit length. From experience, the driver knows the probability distribution of the number of occupied parking spaces she will encounter after initiating cruising for parking before finding a vacant parking space. But she has no information on the realized configuration of occupied parking spaces at the time she commences her outbound journey and receives no information during her journey. Furthermore, she does not exploit information from the pattern of occupied parking spaces that she encounters on her journey to update the probability distribution. Under these assumptions, there is a single rational search strategy. Start searching for parking a distance x from the destination, where x is chosen by the driver to minimize expected trip price, and take the first vacant parking space. There is a meter rate of f per unit time parked. To simplify, traffic congestion is ignored. In-transit travel speed is v , cruising-for-parking speed is s , and walking speed is w with $v \geq 4s \geq 4w$.

Each trip involves an appointment at a specified time. If the driver arrives at the appointment early or late, she encounters a schedule delay cost. The α - β - γ treatment of the value of time is employed; each unit of travel time costs her α (whether in transit, cruising for parking, walking, or visiting at the destination), each

²The space can just as well be an infinite line. The reason I choose the circle is that numerical simulation of the model requires working with a finite space, and working with a circle of finite radius is a manageable way of dealing with the exceptionally unlucky driver who unsuccessfully cruises for parking for a distance exceeding the length of the space.

³Allowing a car that is cruising for parking to turn around after it has passed the destination complicates the algebra without adding insight.

unit of time early costs her β , and each unit of time late costs her γ . Consistent with intuition and the empirical literature, it is assumed that $\gamma > 4\beta$. She decides how long before the specified appointment time, e , to initiate the trip, as well as x , so as to minimize the expected full price of a trip, F , which includes the cost of time on the trip, schedule delay cost, and the curbside parking payment. The expected trip duration equals the expected time spent on the trip to the destination, the time spent at the destination, and the expected time spent on the return trip. The expected time spent on the trip to the destination equals the time driving before initiating cruising for parking, plus the expected time cruising for parking, plus the expected time spent walking from the curbside parking space to the destination. The expected time spent on the return trip equals the expected time spent walking from the destination to the curbside parking space plus the expected return driving time. Expected parking duration equals the time spent at the destination plus the expected time walking from the curbside parking space to the destination and back again.

We now derive expressions for the components of the individual's full trip price. Here and throughout the paper, subscripts denote partial derivatives. z is cruising-for-parking distance, a continuous random variable⁴ whose probability distribution depends on the parking occupancy rate, $Q : g\delta z; Q \beta$. It is assumed that the probability distribution is monotonically decreasing $\delta g_z \delta z; Q \beta < 0$ and that an increase in the occupancy rate decreases the probability that the first vacant parking spot is found within a given distance of cruising for parking $\delta G_Q \delta z; Q \beta < 0$.

Given the informational assumptions, expected cruising-for-parking distance is $Ez\delta Q \beta \int_0^{\delta-x} z g\delta z; Q \beta dz$. Since cruising-for-parking speed is v , expected cruising-for-parking time is

$$ES\delta Q \frac{Ez\delta Q \beta}{v} \tag{1}$$

A driver initiates cruising for parking a distance x prior to reaching her destination. Thus, she may park either before or after reaching her destination. Walking distance between the parking location and the destination (and, on the return journey, between the destination and the parking location) is therefore $|z-x|$. Since walking speed is w , the corresponding expected walking time for the entire journey, EW , is

$$EW = \int_0^x \frac{x-z}{w} g\delta z; Q \beta dz + \int_x^{\delta} \frac{z-x}{w} g\delta z; Q \beta dz \tag{2}$$

Since a driver's destination is a fixed distance δ from the location at which the trip is originated, and since in-transit travel speed is v , the in-transit travel time to the destination prior to initiating cruising for parking is $\delta - x/v$. The return in-transit travel distance equals the distance of the parking space from the trip origin, which equals $\delta - x$. Thus, the expected return

in-transit travel time is $\frac{\delta - x}{v} E z \delta Q \beta = v$, and the total expected

$$E\ell\delta x; Q \beta \frac{2\delta\delta - x\beta E z \delta Q \beta}{v} \tag{3}$$

The expected duration of the trip is the sum of expected cruising-for-parking time, expected walking time, expected in-transit travel time, and visit time. The expected time cost

⁴ In fact, z is a discrete random variable, corresponding to the discreteness of parking locations, but to simplify the algebra I shall treat it as continuous.

⁵ To simplify the algebra, I ignore the possibility that the driver may choose to start cruising for parking as soon as she leaves home. With a very high occupancy rate, this may be optimal. Thus, I am implicitly assuming that this does not occur with the optimal target curbside occupancy rate.

of a trip, $ETTC$, equals the expected duration of a trip times the value of time. Thus,

$$ETTC\delta x; Q \beta \frac{1}{4} \alpha ES\delta Q \beta EW\delta x; Q \beta \beta \ell\delta x; Q \beta \beta \ell \tag{4}$$

To calculate expected schedule delay cost, time is measured relative to the appointment time; viz. $t = 0$ is appointment time. A driver departs at $t = -e$ and arrives at the destination after driving to a distance x before the destination, cruising for parking, and then walking from the parking space to the destination. Thus, the arrival time, a , as a function of x , e , and z is given by

$$a\delta x; e; z \beta \frac{1}{4} - e \beta \frac{\delta - x}{v} \beta \frac{z}{s} \beta \frac{|z-x|}{w} \tag{5}$$

The schedule delay cost is $-\beta a$ with early arrival and γa with late arrival. A driver chooses e such that she arrives early with some probability, and there is always some probability of late arrival.

There are two cases to consider. In the first, the driver who finds a parking space as soon as she starts cruising for parking arrives early. Since cruising-for-parking speed exceeds walking speed, in this case the driver who finds a parking space right at the destination arrives even earlier. In the second case, the driver who finds a parking space as soon as she starts cruising for parking arrives late, but the driver who finds a parking space right at the destination arrives early.⁶ I treat only the first case in the paper, since I judge it to be the more realistic.⁷ The first case applies under the following assumption:

$$A-1: a\delta x; e; Q \beta \frac{1}{4} - e \beta \delta \delta - x \beta = v \beta \delta x = w \beta < 0 \tag{6}$$

The primitive condition under which this case applies is

$$\frac{1}{2} \left(\frac{\alpha w}{\beta \gamma} \right) \geq \frac{\alpha \beta f \beta}{G^{-1} > 2\alpha \beta f \beta - \beta \gamma} > \frac{4}{G^{-1} \frac{\gamma}{\beta \beta \gamma}} \tag{6}$$

The derivation is available in Arnott (2013, Appendix B). Let $M\delta x; e\beta$ denote the z corresponding to on-time arrival, as a function of x and e , i.e. $a\delta x; e; M\delta x; e\beta = 0$. Under A-1, as z increases, the driver arrives increasingly early up to $z = x$, after which she arrives decreasingly early, until she arrives on time at $z = M\delta x; e\beta$, after which she arrives increasingly late. When $z < x$, her walking time is $x - z$ and she arrives early; when $x < z < M\delta x; e\beta$, her walking

time is $\delta z - x\beta = w$ and she arrives early; when $z > M\delta x; e\beta$, her walking time is $\delta z - x\beta = w$ and she arrives late. Thus, $G\delta x; Q \beta$ is the probability that a driver finds a curbside parking space before reaching her destination, and $G\delta x; e; Q \beta$ is the probability that she arrives at her destination early.

Using (5), her expected schedule delay costs are

$$ESDC\delta x; e; Q \beta \frac{1}{4} \beta \int_0^x \frac{\delta}{v} \left(\frac{1}{w} - \frac{1}{s} \right) - x \left(\frac{1}{w} - \frac{1}{v} \right) g\delta z; Q \beta dz + \beta \int_x^{M\delta x; e\beta} \frac{\delta - z}{v} \left(\frac{1}{w} - \frac{1}{s} \right) - \frac{1}{x} \left(\frac{1}{w} - \frac{1}{v} \right) g\delta z; Q \beta dz + \beta \int_{M\delta x; e\beta}^{\delta} \frac{\delta - z}{v} \left(\frac{1}{w} - \frac{1}{s} \right) - \frac{1}{x} \left(\frac{1}{w} - \frac{1}{v} \right) g\delta z; Q \beta dz \tag{7}$$

⁶ It might appear that there is a third case in which both a driver who finds a parking space as soon as she starts cruising for parking and a driver who finds a parking space right at the destination arrive late. Then all drivers would arrive late. But with the cost of time early being lower than the cost of time late, which has been assumed, this cannot be optimal since a driver's expected trip cost would be reduced by departing earlier.

⁷ I have no data to support this judgment. It seems counter-intuitive that a driver who finds a parking space as soon as she starts cruising for parking would arrive late. That it seems so suggests that this case is uncommon.

where

$$M(x, e) = \frac{\delta}{v} \left(\frac{1}{v} \right)^x \left(\frac{1}{w} \right)^{x-1} \left(\frac{1}{s} \right)^{x-1} \quad (8)$$

Her expected curbside parking payment is $f\delta L \beta EW\delta x; Q \beta \beta$.

It is hard to keep track all the cases - cruising past the destination but arriving early, etc. - and hence of the limits on the integrals. Figs. 2 and 3 aim to help. Fig. 2 provides a "location line" for the three relevant cases under A-1: finding a vacant parking space before reaching the destination (case A); finding a vacant parking space after passing the destination and arriving late (case B); and finding a vacant parking space after passing the destination and arriving late (case C). Location is measured relative to the destination. Fig. 3 provides the time lines for those three cases. Time is measured with respect to the start of the appointment scheduled at the destination.

4. Analysis of the model

Before proceeding with the model's analysis, it will be useful to step back for a moment and to consider why equilibrium in the model may be inefficient. Since there is no traffic congestion in the model, there is no congestion externality. The only inefficiency is a parking externality. When a driver occupies a curbside parking space, she reduces the number of vacancies, and hence increases other drivers' costs. Since the externality is linearly proportional to the duration of time a driver occupies a curbside parking space, the appropriate policy instrument to internalize the externality is a meter rate per unit time set at the external cost a driver imposes on others from occupying a curbside parking space per unit time.

When the efficient meter rate is imposed, each driver faces the social cost of her travel, and makes socially efficient decisions not only about how many trips to take but also about how far in advance of her appointment to leave home (e) and when to start cruising for parking (x).

This section looks first at the driver's optimization problem, then at equilibrium, and then at the social optimum and its decentralization.

4.1. A driver's optimization problem

A driver has two decision variables, e the length of time prior to her appointment that she initiates her inbound trip, and x the distance from her destination at which she initiates cruising for parking. She chooses e and x , taking as given the average occupancy rate, Q , and the probability distribution, $g(x); Q \beta$, associated with it, so as to minimize the expected full price of a trip. The expected full price of a trip equals the expected opportunity cost of the trip time, $ETTC$, plus the expected schedule delay cost, $ESDC$, plus the expected curbside parking payment, which equals the per-unit-time meter rate times the expected parking duration, which equals the visit duration plus expected walking time. Thus, the driver's optimization problem is

$$\min_{x, e} F(x, e; Q \beta) = ETTC(x, e) + ESDC(x, e) + Q \beta f\delta L \beta EW\delta x; Q \beta \beta \quad (9)$$

Note that e enters only $ESDC$.

Using (7), the first-order condition with respect to e is

$$\beta G\delta M(x, e) - \gamma \delta (1 - G\delta M(x, e)) = 0 \quad \text{or} \quad G\delta M(x, e) = \frac{\gamma}{\beta \gamma} \quad (10)$$

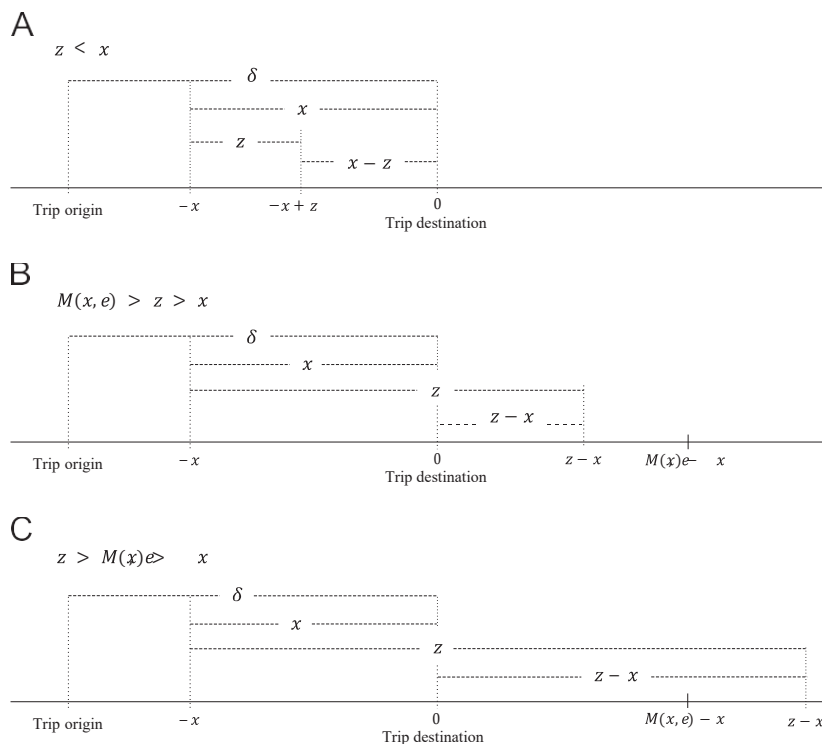


Fig. 2. The location line. The location coordinate is measured with the destination as 0. δ is the distance from the trip origin to the trip destination. Cruising for curbside parking is initiated a distance x before the trip destination (and therefore at location x). Cruising for parking distance is z . If $z < x$, the driver finds the first vacant curbside parking space before reaching her destination, and the walking distance from the curbside parking space to the destination is $x - z$. If $z > x$, the driver finds the first vacant curbside parking space after passing her destination, and the walking distance from the curbside parking space to the destination is $z - x$. $M(x, e)$ is the cruising-for-parking distance corresponding to on-time arrival. If $z < M(x, e)$, the driver arrives at the destination early; if the inequality is reversed, she arrives at the destination late. Under A-1, a driver who parks before her destination arrives early, i.e. $M(x, e) > z$.

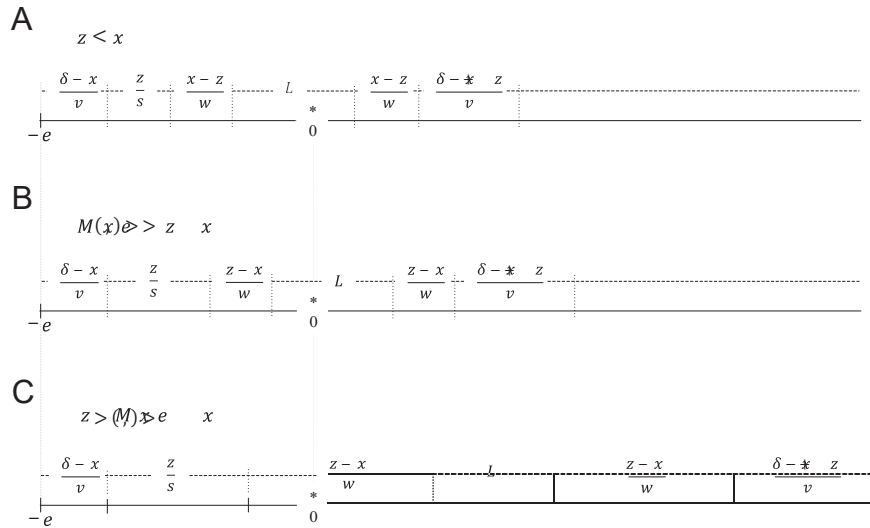


Fig. 3. The time line.

Case A: 1. The time coordinate is measured with respect to the appointment time (marked as 0 below the time line and n above the time line). A driver departs e units of time prior to her appointment. Her trip comprises six segments. The first, in-transit segment lasts until she starts cruising for parking, and, since she travels at speed v during this segment, takes $(\delta - x)/v$ time units. The second, cruising-for-parking segment lasts for the distance she cruises for parking divided by cruising-for-parking speed s : z/s . In the third segment, she walks from her parking location to her destination. Since $z < x$, walking distance is $x - z$, and, since her walking speed is w , this segment occupies $(x - z)/w$ units of time. The fourth segment is the appointment, which is assumed to last L units of time. The fifth segment is the return walk from the destination to the parking location. The last segment is the drive home, a distance $\delta - x$, which takes $(\delta - x)/v$ units of time.

2. Under A-1, in this case she always arrives early, so that her visit at the destination starts before the scheduled appointment time. Her time early is $e - (\delta - x)/v - z/s - (\delta - x)/w$.

Case B: In this case too, the driver always arrives early for her appointment, but she parks beyond her destination so that her walking distance is $(z - x)/w$. Her time early is $e - (\delta - x)/v - z/s - (\delta - x)/w$. $M(x)$ is that value of z that results in on-time arrival (a time early of 0).

Case C: In this case, the cruising for parking distance is sufficiently long that driver parks sufficiently far beyond her destination that she arrives late. Her time late is $(\delta - x)/v - z/s - (\delta - x)/w - e$.

$G\delta M\delta x; e\beta; Q\beta$ is the probability of a driver arriving early. Increasing e by one unit results in expected arrival at the destination a unit time earlier, conditional on being early, with an expected increase in schedule delay cost of $\beta G\delta M\delta x; e\beta; Q\beta$, and expected arrival at the destination a unit time less late, conditional on being late, with an expected decrease in schedule delay cost of $\gamma \frac{1}{\beta} G\delta M\delta x; e\beta; Q\beta$. The driver chooses e to equalize these two magnitudes. Thus, if the unit time late cost is four times the unit time early cost, which roughly accords with the estimated magnitudes, the driver chooses e so that she will arrive early 80% of the time.

The first-order condition with respect to x is

$$ETTC_x - \beta ESDC_x - \beta fEW_x = 0 \tag{11}$$

where, from (1) to (4),

$$ETTC_x = \alpha \frac{4G\delta x; Q\beta - 2}{w} - \frac{2}{v} \tag{12}$$

$$EW_x = \frac{4G\delta x; Q\beta - 2}{w} \tag{13}$$

and from (7) (the derivation is given in Arnott (2013, Appendix A)),

$$ESDC_x = \frac{1}{w} - \frac{1}{v} G\delta x; Q\beta - \beta \frac{1}{w} - \frac{1}{v} G\delta M\delta x; e\beta; Q\beta - G\delta x; Q\beta - \gamma \frac{1}{w} - \frac{1}{v} (1 - G\delta M\delta x; e\beta; Q\beta) \frac{\delta}{v} \tag{14}$$

Combining (11)–(13) gives

$$\alpha \beta \frac{4G\delta x; Q\beta - 2}{w} - \frac{2\alpha}{v} - \beta \frac{1}{w} - \frac{1}{v} G\delta x; Q\beta - \beta \frac{1}{w} - \frac{1}{v} G\delta M\delta x; e\beta; Q\beta - G\delta x; Q\beta - \gamma \frac{1}{w} - \frac{1}{v} (1 - G\delta M\delta x; e\beta; Q\beta) \frac{\delta}{v} = 0 \tag{15}$$

$G\delta x; Q\beta$ is the probability that a driver finds parking before reaching her destination (and therefore arrives early), $G\delta M\delta x; e\beta; Q\beta$ is the probability that she parks beyond her destination and is early, and $1 - G\delta M\delta x; e\beta; Q\beta$ is the probability that she arrives late (and therefore parks beyond her destination). A unit increase in x results in the driver: (a) Walking two units distance further, conditional on finding parking before reaching her destination, and two units of distance less far, conditional on not finding parking until after passing her destination, and therefore in an expected distance of $4G\delta x; Q\beta - 2$ units further, resulting in an increase in expected walking and parking cost of $\frac{4G\delta x; Q\beta - 2}{w}$; (b) Driving in transit two units distance less far, at a saving in in-transit travel time cost of $2\alpha/v$; (c) Conditional on parking prior to reaching her destination (and therefore arriving early), experiencing a decrease in time early cost of $\beta \frac{1}{w} - \frac{1}{v}$; conditional on parking beyond her destination

and arriving early, incurring an increase in time early cost of $\beta \frac{1}{w} - \frac{1}{v}$; and conditional on arriving late (and therefore parking beyond her destination), experiencing a decrease in time late cost of $\gamma \frac{1}{w} - \frac{1}{v}$. Combining (10) and (14) yields

$$G\delta x; Q\beta \frac{\alpha w}{2\alpha \beta f \beta - \beta} \tag{15}$$

The solution to the driver's optimization problem is given by

(8), (10), and (15), which provides three equations in three unknowns, x , e , and M . We write the solutions in compact form as $x = x\delta Q; f\beta$ and $e = e\delta Q; f\beta$. Letting F denote the minimized full trip price, we have that

$$F = \alpha ETTC_x\delta Q; f\beta; Q\beta - \beta ESDC_x\delta Q; f\beta; e\delta Q; f\beta; Q\beta - \beta fEW_x\delta Q; f\beta; Q\beta \tag{16}$$

which is obtained by substituting $x = x\delta Q; f\beta$ and $e = e\delta Q; f\beta$ into (9). The equation gives the technological relationship between the full

Table 1
Comparative static properties of the driver's optimization problem.

	f	Q	α	β	γ	ν	s	w	δ	L
x	–	\uparrow	?	\uparrow	0	–	0	\uparrow	0	0
e	\uparrow	?	?	–	\uparrow	\uparrow	–	–	0	0
$M\delta x; e\beta$	0	\uparrow	0	–	\uparrow	0	0	0	0	0

^a x_Q has the same sign as $f w - \nu - \beta \delta \uparrow \uparrow w - \nu \uparrow$, and e_Q is opposite in sign to x_Q .

trip price and the occupancy rate. We refer to it as the *technology function*.

The comparative static properties of the driver's optimization problem are recorded in Table 1.

The derivations are provided in Arnott (2013, Appendix C). Here we discuss the comparative statics properties of the driver's optimization problem with respect to two variables of particular interest, f and Q .

How x depends on exogenous parameters can be determined from (14). $x_f > 0$. The intuition is as follows. A rise in the curbside meter rate causes a driver to place relatively more weight on reducing expected parking time and relatively less on reducing expected travel time and expected schedule delay. With an infinite curbside meter rate, x is chosen to minimize expected walking time, which is achieved with x such that $G\delta x; Q \uparrow \frac{1}{2}$. But with a finite meter rate, x is chosen so that $G\delta x; Q \uparrow \frac{1}{2}$, which implies initiating cruising for parking farther from the destination than with $G\delta x; Q \uparrow \frac{1}{2}$, since doing so reduces both expected in-transit travel time and expected schedule delay cost. It can also be shown, as intuition suggests, that $x_Q < 0$.

From (10), holding Q fixed, $M \delta x; e$, the distance a driver cruises for parking for on-time arrival, is independent of f . Thus, $M_{x,f} = M_{e,f} = 0$. Since from (8) $M_x < 0$ and $M_e < 0$, and since $x_f > 0$, $e_f < 0$. e_Q is of ambiguous sign, depending on how an increase in Q affects $G_Q \delta z; Q \uparrow = G\delta z; Q \uparrow$ when evaluated at $z \uparrow x$ compared to at $z \uparrow M\delta x; e\beta$.

4.2. Stochastic steady-state equilibrium

Equilibrium is determined as the solution of two equations in two unknowns. The first, the technology function, relates the full trip price to the occupancy rate and is given by (16), but with Q endogenous. The second is the stochastic steady-state condition that, in expectation, the demand for curbside parking time per unit distance–time, which equals the rate at which trips are initiated per unit distance–time times the average curbside parking duration, equals the expected number of occupied parking spaces per unit length. The rate at which trips are initiated per unit distance–time, D , depends on the full trip price: $D \uparrow \frac{1}{2} D\delta F \uparrow$. Expected curbside parking duration equals visit duration plus expected time spent walking from the curbside parking space to the destination and back again, $L \uparrow \frac{1}{2} EW\delta x\delta Q; f \uparrow; Q \uparrow$, where $EW\delta x; Q \uparrow$ is given by (2) and $x\delta Q; f \uparrow$ was obtained in the previous subsection. Thus

$$D\delta F \uparrow \frac{1}{2} L \uparrow \frac{1}{2} EW\delta x\delta Q; f \uparrow; Q \uparrow = Q \uparrow \uparrow \frac{1}{2} Q \uparrow \uparrow \quad (17)$$

Since $F \uparrow \frac{1}{2} D \uparrow \frac{1}{2} Q \uparrow \uparrow \frac{1}{2} L \uparrow \frac{1}{2} EW\delta x\delta Q; f \uparrow; Q \uparrow$, this condition relates the willingness to pay for a trip to the occupancy rate.

It is tempting to interpret (16) as a supply curve and (17) as a demand curve. But the relevant measure of quantity is throughput, not occupancy. Throughput, r , is the steady-state rate at which cars enter and exit the system per unit length, and is related to occupancy according to

$$r \uparrow \frac{1}{2} r \uparrow \delta Q; f \uparrow \uparrow \frac{1}{2} \frac{QP}{\delta L \uparrow \frac{1}{2} EW\delta x\delta Q; f \uparrow; Q \uparrow} \quad (18)$$

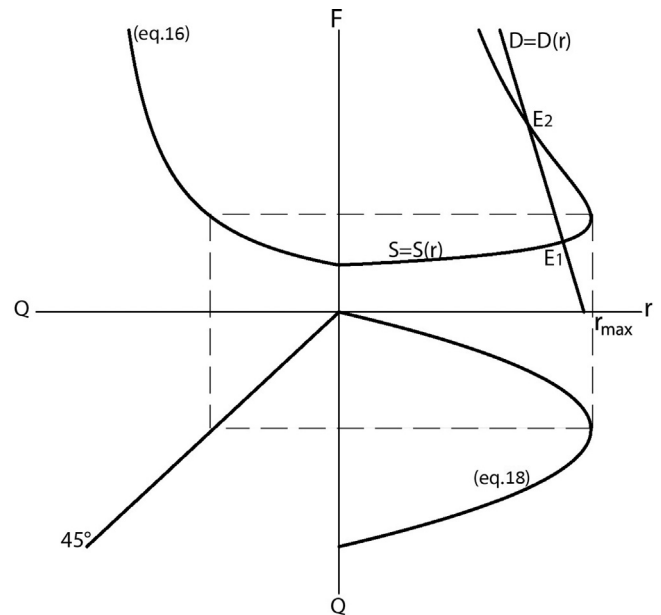


Fig. 4. Graphical procedure for calculating the stochastic steady-state equilibria. Note: The diagram is qualitative. It is not drawn by graphing functions of the model for specific parameter values.

which I refer to as the steady-state condition. Eq. (17) may then be written as $D\delta F \uparrow \frac{1}{2} r$, which is a conventional demand relation.

While I shall work with (16) and (17) in the algebraic analysis, in the diagrammatic analysis I shall focus on $r - F$ space. Fig. 4 presents a four-quadrant diagram. Quadrant II displays (16), which relates F to Q . Quadrant IV displays (18), which relates r to Q .

Quadrant III is the 45-degree line. Quadrant I is the panel of particular interest, since it permits a supply–demand interpretation of (16) and (17). The demand curve is an ordinary demand curve, and slopes downward. The supply curve has upward-sloping and backward-bending portions, and is akin to the supply curve of traffic congestion (Walters, 1961), for which the upward-sloping portion corresponds to congested traffic flow and the backward-bending portion to hypercongested traffic flow. The situation here is almost⁸ completely analogous, except that the congestion occurs in parking rather than in traffic flow. Thus, I shall refer to congested and hypercongested parking. Parking is congested (hypercongested) if the elasticity of expected parking duration with respect to the occupancy rate is less than (greater than) one. To understand why the supply curve has the shape it does, consider the extreme situation where the parking occupancy rate is close to 100%. Expected walking distance from the curbside parking space to the destination is very long, resulting in a very long expected parking duration, and hence a very low throughput.

Fig. 2 displays two equilibria, E_1 and E_2 . Adapting the analysis in Arnott and Inci (2010), which is based on a natural adjustment process, the equilibrium E_1 is congested and stable,⁹ while equilibrium E_2 is hypercongested and unstable. There may be other equilibria as well on the backward-sloping portion of the supply curve. All such interior equilibria are hypercongested, and alternate between stable and unstable. There is always a gridlock

⁸ The analogy is not complete since here, unlike in Walters' (1961) analysis of traffic congestion where the individual driver chooses only trip frequency, the individual driver chooses not only trip frequency but also x and e .

⁹ It is possible that the demand curve intersects the supply curve only on its backward-sloping portion. In this case, the lowest equilibrium, which corresponds to E_1 , is hypercongested and stable.

Table 2
Comparative static properties of the stable, congested equilibrium, E_1 .

	f	η
Q	–	\uparrow
r	$?$ ^a	\uparrow
F	$?$ ^a	\uparrow

^a The signs of r_f and F_f are opposite.

equilibrium as well, at an infinite price and zero throughput.¹⁰ The optimum, decentralized via the optimal meter rate, corresponds to an equilibrium of type E_1 .

The comparative static derivatives are complicated since there are three potential channels through which a change in an exogenous parameter affects an endogenous variable, through x , e , and Q . Arnott (2013, Appendix D), derives the comparative static derivatives with respect to f and η , where η is a demand shifter, which I term “demand intensity”, with higher η corresponding to a higher demand curve. The comparative static properties of the stable, congested equilibrium, E_1 , are given in Table 2.

The effects of an increase in demand on the stable, congested equilibrium can be seen from Fig. 4. The demand curve shifts out, and the supply curve does not change position. Thus, the increase in demand unambiguously increases both throughput and the full price of a trip. Furthermore, since throughput and the occupancy rate are positively related when parking is congested, the increase in demand unambiguously increases the occupancy rate.

The effects of an increase in the meter rate on the stable congested equilibrium are more complicated. The meter rate increase shifts the supply curve but has no effect on the demand curve. The immediate effect of a unit increase in the meter rate is to shift the supply curve up by the expected parking duration (which is analogous to a congestion toll shifting the supply curve up by the amount of the toll).¹¹ But (unlike in the conventional analysis of the congestion toll), there are other effects as well due to the margins of choice, departure time, e , and the distance before the destination that cruising for parking is initiated, x . The increase in the meter rate causes the driver, in her choice of e and x , to put more weight on parking costs relative to the other types of costs, which results in a decrease in her expected parking duration. The decrease in expected parking duration in turn decreases the occupancy rate, which in turn affects expected parking duration. The overall effect of these changes on throughput and the full price is ambiguous. If the proportional fall in the occupancy rate is less than the proportional fall in parking duration, throughput rises (see (18)) and the full price falls (since equilibrium moves down the stationary demand curve). If, furthermore, the proportional rise in the meter rate exceeds the proportional fall in the occupancy rate, meter revenue rises. Thus, one possibility is a double dividend result, in which the gain in social surplus from raising the meter rate exceeds the increase in the meter revenue collected.

¹⁰ As the occupancy rate goes to one, a driver expects to cruise for parking for an infinite period of time, walk an infinite distance between the parking space and the destination, and park for an infinite duration. The throughput demanded equals zero since the expected full price is infinite. The throughput supplied is zero since the expected turnover rate, the reciprocal of the expected parking duration, equals zero.

¹¹ If this were the end of the story, the rise in the supply curve with no change in the demand curve would result in a decrease in throughput, associated with which would be a decrease in consumer surplus. If the meter rate is below the optimal level, increasing it increases social surplus, which requires that the increase in meter revenue more than offset the decrease in consumer surplus.

4.3. Social optimum

The social optimum occurs where the marginal social cost of throughput equals the marginal social benefit. Here, since there are no externalities on the demand side, the marginal social benefit at a given level of throughput is given by the corresponding point on the demand curve. Calculating marginal social cost is complicated by the fact that per driver cost depends on the occupancy rate rather than throughput. We proceed as follows. First, we define minimized total cost as a function of throughput:

$$T\hat{C}(r, \lambda) = \min_{x, e; Q} [r\hat{E}TTC(x); Q\hat{P}\hat{E}SDC(x); e; Q\hat{P}]$$

$$s.t.: Q\hat{P} - r\delta L\hat{P}EW(x); Q\hat{P} \geq 0 \tag{21}$$

where $EW(x)$; $Q\hat{P}\hat{E}TTC(x)$; λ , and $ESDC(x)$; $e; Q$ are given by (1)–(4), (7), and (8). Then

$$M\hat{S}C(r) = \frac{dT\hat{C}(r)}{dr}$$

The above procedure calculates the direct control total cost function, assuming that the planner chooses x and e . But it is the individual driver and not the planner who chooses x and e . Thus, the total indirect control cost function should be calculated, taking into account that x and e are chosen by individual drivers, with the planner having only indirect control of x and e , through the meter rate. It turns out, however, that this complication is immaterial since, when the planner chooses the meter rate optimally, drivers choose the socially optimal x and e . Thus, to simplify the analysis, I solve for the direct control social optimum, and then for the optimal meter rate that decentralizes it.

Where $X(r)$ is the total social benefit derived from r trips per unit area-time (the area under the demand curve), the direct control social welfare optimization problem is

$$\max_{r, \lambda} [X(r) - r\hat{E}TTC(x); Q\hat{P}\hat{E}SDC(x); e; Q\hat{P}]$$

$$s.t.: Q\hat{P} - r\delta L\hat{P}EW(x); Q\hat{P} \geq 0 \tag{22}$$

where λ is the shadow price on this form of the steady-state condition. The first-order conditions are

$$r : X' - ETTC - ESDC - \lambda\delta L\hat{P}EW(x) = 0$$

$$Q : -r\hat{E}TTC_Q\hat{P}ESDC_Q\hat{P}\lambda\delta P - r\delta EW_Q\hat{P} = 0$$

$$x : -r\hat{E}TTC_x\hat{P}ESDC_x\hat{P} - \lambda r\delta EW_x\hat{P} = 0$$

$$e : -r\hat{E}ESDC_e\hat{P} = 0 \tag{23}$$

Note four things. First, it is evident from the first-order condition with respect to r that λ is the shadow price of a curbside parking space per unit time. Second, the first-order condition with respect to e is the same as the corresponding driver's first-order condition, (10). Third, from (11), the driver's first-order condition with respect to x is the same as the corresponding first-order condition for the social optimum when the meter rate is set equal to the shadow price of parking. Thus, as claimed, drivers make socially efficient decisions with respect to both x and e when the meter rate is set equal to the shadow price of parking. The intuition is straightforward. There is only one externality in the model, the parking externality. When this is internalized, drivers make socially efficient decisions. Fourth, the problem is decomposable. The first step entails calculating $T\hat{C}(r)$ per (19), the second step entails maximizing $X(r) - T\hat{C}(r)$ with respect to r .

From the first-order condition with respect to Q , the shadow price of a curbside parking space equals

$$\lambda = \frac{r\hat{E}TTC_Q\hat{P}ESDC_Q\hat{P}}{P - r\delta EW_Q\hat{P}} \tag{24}$$

The interpretation of this shadow price as the parking externality cost requires some care. First, the λ in (22) is exactly the same as the λ in (19), since the externality is a production externality.

Second, the externality operates through Q and not directly through r . Instead, a marginal increase in r affects Q , and the marginal increase in Q generates the external costs by increasing all drivers' trip cost. Third, while there are two values of Q that solve the steady-state condition for a given r , one associated with hypercongested parking, the other with congested parking, in determining the social optimum only the congested value of Q is relevant. Thus, we may express how the social cost minimizing Q varies with r , per (19), as $Q \approx \frac{1}{4} Q^n \delta r$, and similarly we may write $e \approx \frac{1}{4} e^n \delta r$ and $x \approx \frac{1}{4} x^n \delta r$. Then total cost can be rewritten as

$$TC \approx \frac{1}{4} r Z \delta x^n \delta r; e^n \delta r; Q^n \delta r$$

where

$$Z \delta x^n \delta r; e^n \delta r; Q^n \delta r \approx \frac{1}{4} ETTC \delta x^n \delta r; Q^n \delta r \approx \frac{1}{4} ESDC \delta x^n \delta r; e^n \delta r; Q^n \delta r$$

Since the derivatives with respect to $x^n \delta r$ and $e^n \delta r$ equal 0 via the Envelope Theorem

$$MSC \approx \frac{1}{4} Z \delta x^n \delta r; e^n \delta r; Q^n \delta r \approx \frac{1}{4} Z \delta x^n \delta r; e^n \delta r; Q^n \delta r$$

where, from the steady-state condition, $r Q^n \delta r \approx \frac{1}{4} \lambda \approx EW$. The first term on the right-hand side is the marginal driver's trip cost, and the second is the parking externality cost, which is analogous to the familiar congestion externality cost. Raising throughput by 1 unit increases the occupancy rate by $Q^n \delta r$, which raises each inframarginal driver's cost by $Z \delta x^n \delta r; e^n \delta r; Q^n \delta r$, and inframarginal drivers' total cost by $Z \delta x^n \delta r; e^n \delta r; Q^n \delta r$.

The social optimum is decentralized simply by setting the meter rate equal to λ .

In the standard diagrammatic analysis of the traffic congestion externality, the marginal congestion externality cost equals the vertical distance between the marginal social cost of a trip and the user cost. Here, a driver's user cost depends on her choice of x and e , which depends on the meter rate. Define the user cost function when the meter rate is set equal to the value of λ , evaluated at the social optimum, λ^n , to be

$$UC \approx \frac{1}{4} r ETTC \delta x^n \delta r; \lambda^n; \lambda^n; Q^n \delta r; \lambda^n \approx \frac{1}{4} ESDC \delta x^n \delta r; \lambda^n; \lambda^n; e^n \delta r; \lambda^n; Q^n \delta r; \lambda^n$$

where $Q^n \delta r; \lambda^n$ corresponds to the congested solution to the steady-state condition when the meter rate is set equal to λ^n .

Fig. 5 presents the social optimum diagrammatically in a form familiar from the analysis of congestion pricing. The social optimum occurs at the intersection of the demand curve and the marginal social cost curve, msc . The curve uc is the user cost curve when the meter rate is set at the optimal level, per the above definition. When the meter rate is set at the optimal level, the supply curve is the user cost curve shifted up by $\lambda^n \approx \frac{1}{4} EW$. And the equilibrium when the meter rate is set at the optimal level coincides with the optimum.

The comparative static derivatives of the social optimum with respect to η , which we term demand intensity and which corresponds to an outward shift in demand, are¹²

$$\frac{dr}{d\eta} > 0; \frac{dQ}{d\eta} > 0; \frac{dx}{d\eta} ?; \frac{de}{d\eta} ?; \frac{dM \delta x; e}{d\eta} > 0; \frac{d\lambda}{d\eta} > 0; \frac{dQ^n}{d\eta} > 0$$

Note that the increase in η affects the driver's choice of x and e through an increase in both Q and f .

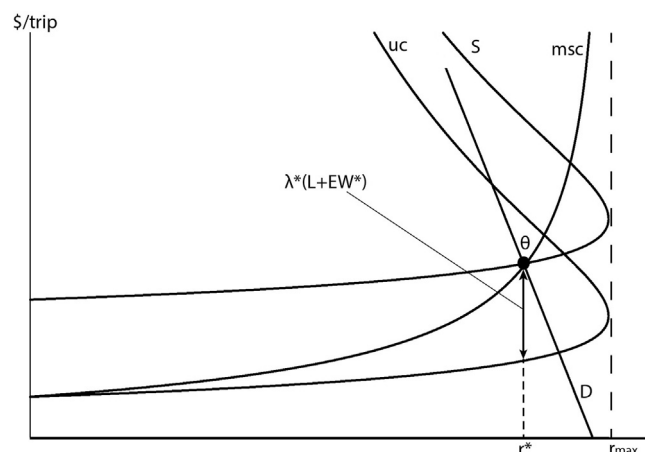


Fig. 5. Decentralization of the social optimum. Note: The diagram is qualitative. It is not drawn by graphing functions of the model for specific parameter values.

One comparative static result is central:

Proposition 1. *The socially optimal target curbside parking occupancy rate is increasing in demand intensity.*

This result is central since it implies that the search for a universal socially optimal curbside parking occupancy rate is a will o' the wisp. The intuition is straightforward. A shift upwards in the marginal social benefit curve in Fig. 3 implies an increase in optimal throughput. Furthermore, since it is optimal for parking to be congested rather than hypercongested, the rise in optimal throughput is associated with an increase in the occupancy rate.¹³

Most other comparative static derivatives are ambiguous in sign.¹⁴

For one thing, from (21) most contain terms in $ETTC_{Q0}$, $ESDC_{Q0}$, and/or EW_{Q0} , each of which depends on G_{Q0} , about which no assumption has been made.

It is tempting to argue, on the basis of the central result, that in real-world situations, the optimal occupancy rate should be higher during peak than in off-peak hours, or at more rather than less congested locations. Those conjectures may prove sound.¹⁵ But I

¹³ This result was derived under A-1, that a driver who finds a vacant curbside parking space as soon as she starts cruising for parking arrives at her destination early. It is also valid under the alternative assumption, that a driver who finds a vacant curbside parking space as soon as she starts cruising for parking arrives at her destination late. Under both assumptions optimal throughput is increasing in demand intensity, and the optimal occupancy rate is increasing in optimal throughput, which are all that is needed for the result.

¹⁴ Rather obviously, a doubling of all monetary variables has no effect on the optimum r , Q , x , e , and M , but causes a doubling of F .

¹⁵ The comparative static result that an increase in demand intensity leads to a higher optimal curbside occupancy rate can likely be generalized somewhat beyond the specific model considered in the paper. The result that an increase in demand intensity leads to higher optimal throughput applies in any generalization for which the marginal social cost curve and the demand curve can be portrayed as in Fig. 3. From the literature on the bottleneck model, we know that, with identical individuals, the reduced form of the morning rush-hour optimum in this paper's model can be so displayed (Arnott et al., 1991). Taking as given the throughput over the entire rush hour, N , using optimal control theory one can solve for the optimal time pattern of departures over the rush hour, and the corresponding total social cost, $TC^0(N)$, and hence the marginal social cost, $MSC^0(N)$. Furthermore, since, in the corresponding decentralized social optimum, in which the optimal time-varying meter rate is applied, all drivers face the same trip price (which equals $MSC^0(N)$), the demand function, $D(N)$, is well defined. The socially optimal level of throughput over the rush hour, N^0 , is then obtained as the point of intersection of $MSC^0(N)$ and $D(N)$, from which the optimal time-varying meter rate may be backed out. In steady state, the expected occupancy rate remains constant over time. In the morning rush hour, however, the expected occupancy rate varies over the rush hour. I conjecture, but have not proved, that an increase in demand intensity causes the occupancy rate as a function of time to shift upward. The same argument applies when there are constraints on the form of the time-varying meter rate, for example, that it be

¹² The derivations may be obtained from the author upon request.

think it is premature to base policy on them. First, the real world is more complex than the model here, including intra-day traffic dynamics, heterogeneity of drivers and street space, and two-dimensional rather than one-dimensional search. Second, the central comparative static result relates to a change in a single exogenous variable, η , whereas, in the real world, exogenous variables typically do not change one at a time; for example, more congested locations may have a systematically lower or higher curbside parking density, depending on curbside parking policy, and peak-period drivers may have a higher value of time than off-peak drivers. Third, there are scale effects in cruising for parking.¹⁶ Nevertheless, since negative results generalize from the more specific to the more general, one may safely conclude that, in real-world situations, *the optimal target curbside occupancy rate is not constant over time and space.*

5. Directions for future research

In discussing directions for research, I have in mind what would need to be done to extend the model for use in practical applications, such as SFpark.

Incorporating traffic congestion: The interaction between curbside parking and traffic congestion is practically important, and should be considered in any model that aims to derive a realistic optimal target curbside parking occupancy rate. In a series of related papers, Arnott and Inci (2006, 2010), Arnott and Rowse (2009, 2013), and Arnott et al. (2013), Arnott, Inci, and Rowse have developed a sequence of related models that treat the interaction between curbside parking, garage parking, and traffic congestion in a isotropic, two-dimensional area in steady state.¹⁷ The model of this paper could be augmented to include the interaction between curbside parking and traffic congestion in the same way as was done in Arnott and Inci (2006).

With traffic congestion added, the optimal amount of curbside to allocate to parking (optimal curbside parking capacity) can be determined both when traffic congestion is efficiently priced and when it is not, as was done in Arnott et al. (2013) with their less sophisticated treatment of curbside parking. For each level of

(footnote continued)

constant over the morning rush hour or that there be a single step, as long as the level of the meter rate can be varied so that the common trip price equals $MSC(N)$. The same argument also applies when the paper's model is generalized to include traffic congestion, in which case decentralizing the optimum requires not only an optimal time-varying meter rate but also an optimal time-varying toll. Furthermore, this argument can be extended to situations in which there are constraints on the forms of both the time-varying meter rate and congestion toll, as long as the level of either can be varied so that the common trip price equals $MSC(N)$.

Intuitively, for the steady state, the result should apply with user heterogeneity with respect to α , β , γ , δ , L , and t^0 , with each user group having a separate demand function, when each group's demand intensity increases. For morning rush-hour traffic dynamics, the situation is more complicated since different user groups depart in a particular sequence. With user heterogeneity, the issue arises as to whether the optimal time-varying meter rate can be differentiated according to user type.

¹⁶In the analysis of traffic congestion, it is often assumed that travel time depends on the ratio of flow to capacity flow (or of traffic density to capacity density), so that a doubling of both has no effect on travel time. In the analysis of curbside parking search, the analogous result would be that, holding fixed the occupancy rate, a doubling of the density of cars cruising for parking and of curbside parking capacity has no effect on cruising-for-parking time. But this is incorrect. It has no effect on the expected number of curbside parking spaces searched before a vacant spot is found, but since the distance between parking spaces is halved, expected cruising-for-parking distance and cruising-for-parking time are halved.

¹⁷The papers provide an unsophisticated treatment of curbside parking search. Either parking is unsaturated everywhere, in which case curbside parking search costs are zero, or parking is saturated everywhere, in which case expected curbside parking search time equals the stock of cars cruising for parking per unit area divided by the turnover rate per unit area.

demand, social surplus is solved for as a function of curbside parking capacity, with account being taken that drivers decide on trip frequency, as well as x and e , taking the occupancy rate and meter rate as fixed, so as to maximize their private surplus.

Garage parking: Arnott and Rowse (2009) added private garage parking to the Arnott-Inci model of downtown parking and traffic congestion. That paper ignores the costs of searching for parking inside parking garages, and provides two treatments of private garage location and costs. In the simpler treatment, garage parking is provided continuously over space at constant cost and priced at this cost (Bertrand competition). In the more sophisticated model, garages are discretely spaced due to economies of scale in garage construction, and consequently have market power. Garage parking could be introduced into the model of this paper in either of these two ways. Since SFpark is adjusting parking prices so as to achieve a target occupancy rate not only curbside but also in public parking garages, the model would be more useful if it were extended to treat search for parking inside parking garages.

Rush-hour dynamics: One earlier paper, Arnott et al. (1991), and several recent papers (Qian et al., 2012; Zhang et al., 2008, 2011) have been written that extend Vickrey's bottleneck model of rush-hour traffic congestion to include parking. The papers provide different simplified treatments of parking and cruising for parking. The model of this paper, with its more sophisticated treatment of parking, could be extended relatively straightforwardly to treat rush-hour traffic and parking dynamics via the bottleneck model. Doing so would permit investigation of whether the target curbside parking occupancy rate should differ according to the stage of the rush hour.

Two-dimensional space: This paper had two main goals. The first was to develop a structural model of curbside parking that has the potential of being extended to the point where it can be applied in practical policy contexts. The second was to make the general point that there is no universal optimal target curbside parking occupancy rate. The paper has, I think, achieved these modest goals. But the model is still far from practical application. One of its most obvious deficiencies is that it almost trivializes the curbside parking search problem by treating it as one-dimensional. But practically curbside parking search is two-dimensional, and two-dimensional parking search is much more difficult to treat satisfactorily than one-dimensional parking search. A first step in analyzing two-dimensional parking search is to model it in an isotropic space (an infinite plain or the surface of a sphere). A driver's optimal parking search strategy (e.g., drive to the destination block, and cruise around the block until a parking space opens up) depends on other drivers' search strategies, which suggests that there may be multiple equilibria.

Anisotropic space: The monocentric city is perhaps the simplest interesting anisotropic space. Arnott et al. (1991) and Anderson and de Palma (2004) have analyzed curbside parking in the monocentric city model. Drivers park from the CBD either inwards or outwards depending on the meter rate structure. Parking inwards is more efficient since it concentrates the distribution of arrivals, conditional on the distribution of departures. While analysis of curbside parking in the monocentric city model generates important general insight, for policy purposes what is of interest is the actual street network, which requires downtown traffic microsimulation models to deal with. Thus, an important topic on the research agenda is to strengthen these models' parking modules, for instance by accommodating heterogeneity in search strategy.

Heterogeneity: It remains to be seen how important driver heterogeneity is in determining the optimal target curbside parking occupancy rate.

Estimating the ϕ z; \mathcal{Q} function: SFpark is collecting comprehensive data on the occupancy histories of public parking garages and individual parking meters. These data are insufficient to measure

social surplus. To measure social surplus, it is necessary to relate aggregate travel costs to the observables, such as the curbside occupancy rate and the distribution of curbside parking times. This paper's model indicates that the form of $g(z)$; Q is central to this relationship. All previous theoretical work has assumed this function to be negative exponential (where z is treated as continuous) or binomial (where z is treated as discrete). But Levy et al. (2013) persuasively demonstrate that this assumption may lead to a severe downward bias in the estimation of curbside parking search time as a function of the occupancy rate. The form of this function depends on the curbside parking search strategy. Determining its form analytically under alternative parking search strategies appears to be very difficult. A more promising avenue is to estimate it from microsimulation, for first isotropic networks, then different idealized types of anisotropic networks, and then real networks.

Curbside capacity, parking time limits, and the nonlinear pricing of curbside parking: Recent policy discussion has focused on cashing out free and heavily subsidized curbside parking by raising meter rates until curbside occupancy rate targets are met, with little attention being paid to other aspects of curbside parking policy, such as curbside parking capacity, curbside parking time limits, and the nonlinear pricing of curbside parking. In an extended calibrated numerical example, Arnott et al. (2013) illustrate that, in heavily congested downtown areas where garage parking is available, it may be optimal to eliminate curbside parking altogether, at least on the most congested streets, especially when curbside parking is substantially underpriced. When drivers differ in visit length and the value of time, when curbside parking is underpriced, and when garage parking is available, Arnott and Rowse (2013) show that curbside parking time limits both decrease cruising for parking and increase the parking turnover rate, both of which make shopping downtown more attractive. Curbside parking time limits along with a curbside parking meter rate generate a form of non-linear pricing of curbside parking. Today's hi-tech parking meters permit the implementation of arbitrary non-linear curbside parking pricing.

Endogenous parking duration: To the extent that the driver has the ability to do so, she chooses a *visit* duration such that the marginal benefit of extending the visit a unit of time equals the marginal cost. The marginal cost is the meter rate plus the opportunity cost of time, and the marginal benefit is independent of the occupancy rate. If the meter rate is raised (lowered) to reach the target occupancy rate, visit duration is shorter (longer).

The model assumes that the driver arranges only one appointment or activity per trip. But as the full price of a trip increases, it is more likely that the driver chooses to undertake more than one activity per trip. Suppose, for the sake of argument, that she has decided to take a trip, and to arrange two appointments per trip. She then has the choice between parking separately for each appointment or parking only once, and then walking from the first to the second appointment. The choice is made by backward induction. After the first appointment is over, depending on where she found parking, she decides between walking directly to her second appointment, returning to her parking spot and then searching for parking for her second appointment, and abandoning her second appointment, on the basis of expected cost. She makes her decisions whether to take the trip at all, and conditional on taking the trip, chooses x for the first appointment and e , so as to maximize her surplus, which depends on the probabilities of these three outcomes.

6. Conclusion

Policy makers are coming to recognize the importance of parking policy in the management of downtown auto congestion. One aspect of parking policy that has recently received considerable attention is

efficient curbside parking pricing. The general rule is that the market-clearing price is the efficient price. This rule applies to curbside parking, but because of the stochasticity associated with curbside parking entry and exit, its implementation would require responsive pricing, in which the meter rate on a particular block at a particular time of day would depend on the particular realization of the stochastic process. Such pricing would be informationally demanding, hard to implement, and annoying to drivers. A more practical policy is to set meter rates *ex ante*. The efficient *ex ante* meter rate would balance the efficiency cost of having the meter rate below its market-clearing level when realized demand is high (cruising for parking costs) and the efficiency cost of having the meter rate above its market-clearing level when demand is low (unutilized curbside parking capacity).

How should the efficient *ex ante* meter rate on a particular block and for a particular period of the day be determined in practice? Donald Shoup has advocated setting block- and time-period specific meter rates so that a common average curbside parking occupancy rate is achieved. The cities of San Francisco and Los Angeles are implementing a variant of Shoup's proposal on an experimental basis (SFpark and LA Express Park, respectively).

This paper developed a simple, structural model of curbside parking to investigate the theoretical basis for an optimal target curbside parking occupancy rate rule. Parking takes place on the outside of a circle, and only the (stochastic) steady state is analyzed. The analysis contained three elements. The first solved for drivers' optimal strategy in cruising for parking, taking as given the probability function for the number of curbside parking spaces searched before a vacant space is found, which depends on the curbside occupancy rate, as well as the meter rate. The second solved for steady-state equilibrium in which, in expectation, occupied parking spaces per unit length (the occupancy rate times the density of parking spaces per unit length) equal throughput per unit length times expected parking duration (visit duration plus expected time walking between the parking space and the destination). In equilibrium, there is an uninternalized parking externality since each driver increases the curbside occupancy rate, increasing other drivers' average distance cruising for parking and walking between the parking and destination locations. And the third solved for the social optimum. Analogously to Walters' well-known diagrammatic analysis of steady-state traffic congestion (1961), at the optimum the marginal social cost of throughput equals the marginal social benefit, and the optimum can be decentralized by imposing a meter rate that internalizes the parking externality, such that a driver faces the marginal social cost of a trip. The model's central comparative static result is that the optimal curbside parking occupancy rate is higher, the higher is the level of demand intensity. This suggests the conjecture that *in practice* the optimal occupancy rate is higher at busier locations and at busier times.

The paper provides the conceptual basis not only for determining the optimal target curbside parking occupancy rate but also for undertaking welfare analysis of other policies related to curbside parking. Much remains to be done, however, in extending the model in the direction of realism, before it can usefully be implemented in specific policy contexts.

SFpark is gradually adjusting meter rates by block and time of day until target curbside occupancy rates are achieved. Comprehensive data are being collected on the occupancy experience of every parking meter and every public parking garage in the programs. But to undertake welfare analysis, it is necessary to relate occupancy rates to driver costs, including cruising-for-parking time costs, walking costs, and schedule delay costs. One approach is to collect the data needed to estimate these relationships directly. An alternative approach is to apply a structural model, such as an extended version of this paper's model. A crucial

element of both approaches is to estimate cruising-for-parking search strategies.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found in the online version at <http://dx.doi.org/10.1016/j.ecotra.2014.07.001>.

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CRUISING FOR PARKING AROUND A CIRCLE

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Abstract

There has recently been considerable interest in cruising for curbside parking as a major contributor to traffic congestion in the downtown areas of major cities. This paper focuses on the expected search time for a curbside parking space. The literature has employed three different approaches to estimate expected cruising-for-parking time: direct measurement, inference based on the equilibrium condition that (for the marginal parker) the expected cost of curbside parking equals the expected cost of garage parking, and inference based on the observed occupancy rate of curbside parking and an assumed statistical relationship between expected cruising-for-parking time and the curbside parking occupancy rate. The last approach typically obtains estimates of expected cruising-for-parking times that are lower, and with high occupancy rates much lower, than those estimated using the other two approaches. This paper takes a step towards resolving this inconsistency by demonstrating, through computer simulation of cars cruising for parking around a circle in stochastic steady state, that an approximating assumption in the derived statistical relationship between expected cruising-for-parking time and the curbside parking occupancy rate leads to underestimation of average cruising-for-parking time, and at high occupancy rates very considerable underestimation. The paper also identifies several "effects" that contribute to the approximating assumption being an increasingly poor one as the occupancy rate increases.

Keywords curbside parking, stochastic simulation, Poisson process, steady state

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Highlights

- Stochastic simulation of cruising for curbside parking round a circle.
- Previous work has used the approximation that curbside vacancies occur independently.
- That approximation leads to severe underestimation of mean cruising-for-parking times.
- Identifies four effects that lead to underestimation.
- Results point to potential benefits of responsive curbside parking pricing.

Cruising for Parking around a Circle¹

The pioneering work of Donald Shoup (2005) has stimulated considerable discussion of cruising for curbside parking as a major contributor to traffic congestion in the downtown areas of major cities. The literature contains estimates that the proportion of cars traveling on downtown city streets during the business day that are cruising for parking is 30% or even higher. Such estimates are not obtained from sidewalk observation since cars that are cruising for parking cannot be distinguished from cars in transit, but are instead obtained either by following a sample of cars or through model-based inference.

The density of cars cruising for parking in the downtown area is related to the rate at which cars in transit in the downtown area start cruising for parking and the expected search time of a car that searches for parking. This paper focuses on this expected search time. The literature has employed three different approaches to estimate expected cruising-for-parking time: direct measurement, inference based on the equilibrium condition that (for the marginal parker) the expected full price (which equals the money price plus the opportunity cost of time) of curbside parking equals that of garage parking,

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and inference based on an assumed statistical relationship between expected cruising-for-parking time and the curbside parking occupancy rate.

Most of the studies that employ direct measurement of cruising for parking are reviewed in Shoup (2005, Chapter 11). There are two reasons to be skeptical of the results. The first is that it is difficult to identify cars that are cruising for parking. One approach is to delineate a study area, follow random cars that enter the study area, identify them as searching for parking if they park curbside in the study area, and measure their travel times within the study area. This approach fails to identify cars that are indeed searching for curbside parking in the study area but end up parking either outside the study area or in a parking garage. It also fails to identify when cars that park curbside in the study area initiate cruising for parking. The second reason to be skeptical of the results is that the study areas were not randomly selected, but were chosen instead because cruising for parking was perceived to be a severe problem there.

The second approach to estimate expected cruising-for-parking time was employed in Arnott and Rowse (2009, 2013). It is based on a model in which risk-neutral drivers choose between ubiquitous curbside and garage parking. Curbside parking is fully saturated, so that a car enters a curbside parking space immediately after it is vacated²,

² Arnott and Rowse assumed that each car cruising for parking experiences a vacant parking space according to a Poisson process with a rate equal to the turnover rate of parked cars per unit area divided by the stock of cars cruising for parking per unit area. When parking durations are negative exponentially distributed, the authors conjecture that the simulation model of this paper has this property in the limit as the expected occupancy rate approaches 100%.

while garage parking can be obtained without search. In equilibrium, drivers sort between curbside and garage parking such that their full prices are the same for "marginal" parkers -- those who are indifferent between curbside and garage parking. In most cities except those in the Netherlands, curbside parking is considerably cheaper than garage parking. An alternative statement of the parking equilibrium cost condition is then that, for a marginal parker, the curbside cruising-for-parking time cost equals the savings in the money cost from parking curbside. In the case of identical individuals, the expected cruising-for-parking time equals the savings in the money cost from curbside parking divided by the common value of time. Consider an example with identical drivers in which the parking duration is one hour, the meter rate is³ \$1.000/hr, the one-hour garage parking fee is \$10.00, and the value of time is \$30.00/hr. Since the saving in the money cost of curbside parking is \$9.000, the equilibrium expected cruising-for-parking time is 0.3000 hrs.

The third approach considers a situation in which curbside parking is not saturated but is instead described by an expected occupancy rate. The central assumption is that the probability that each curbside parking space is occupied equals the expected occupancy rate, independent of history and of the occupancy status of neighboring curbside parking spaces. We term this the *binomial approximation*. It generates a geometric distribution for the number of parking spaces searched before finding a vacant space (including the vacant space). The number of parking spaces searched corresponds to the number of balls drawn from an urn with replacement (or with an infinitely large number of balls) before a

³ Throughout the paper, numbers are presented to the fourth significant digit.

"vacant" ball is drawn. Let q denote the probability that a ball is labeled "occupied", so that $1 - q$ is the probability that a ball is labeled "vacant". The probability of finding the first vacant space on the first draw (i.e., the first parking space searched) is $1 - q$; the probability of finding the first vacant space on the second draw is $q(1 - q)$, which is the probability that the first space searched is occupied times the probability that the second space searched is vacant, etc. The expected number of parking spaces searched before finding a vacant space (including the vacant space) is⁴ $1/(1 - q)$. Thus, the expected numbers of parking spaces searched (including the vacant space) with curbside parking vacancy rates of 20%, 10%, 5%, and 1% are 5, 10, 20, and 100 respectively. Expected cruising-for-parking time can then be obtained by applying estimates of the average distance between parking spaces and of cruising-for-parking speed. As an example, assume that the distance between curbside parking spaces is 21.12 ft (1/250 mi) and that cruising-for-parking speed is 8.000 mph. Then the average cruising-for-parking time between parking spaces is 1/2000 hrs or 1.800 seconds. Shoup (2006) proposed⁵ that curbside meter rates be set to achieve a curbside parking occupancy rate of 85%. Under

⁴ Let the expected number of draws before drawing a vacant ball (including the draw with the vacant ball) be S . $S = (1)(1 - q) + (2)[q(1 - q)] + (3)[q^2(1 - q)] + \dots = (1 - q)\{1 + 2q + 3q^2 + \dots\}$. Multiplying both sides by q yields $qS = (1 - q)\{q + 2q^2 + 3q^3 + \dots\}$. Subtracting qS from S yields $(1 - q)S = (1 - q)\{1 + q + q^2 + \dots\}$. Since the value of the infinite sum in the curly brackets is $1/(1 - q)$, $S = 1/(1 - q)$.

The variance, skewness, and Fisher kurtosis of the geometric distribution are $q/(1 - q)^2$, $(1 + q)/q^{1/2}$, and $6 + (1 - q)^2/q$ (Wikipedia: Geometric distribution).

⁵ Shoup's work has stimulated a number of downtown parking experiments. The best known is SFpark. The City of San Francisco has been adjusting curbside meter rates by block and by time of day to achieve a target curbside parking occupancy rate. Shoup (2006) originally proposed a target curbside parking occupancy rate of 85%. The City has been adjusting this rate by block and time of day to achieve what it judges to be optimal rates. They vary substantially but the average is considerably lower than 85% (Pierce and Shoup, 2013).

the binomial approximation and the above parameter assumptions, applying the Shoup rule would generate expected cruising-for-parking time of only 12.00 seconds ($1.800 \times 1/(1 - 0.8500)$).

All the recent papers that derive the expected cruising-for-parking time from the occupancy rate, including Arnott and Rowse (1999), Anderson and de Palma (2004), Geroliminis (2015), and Du and Gong (2016), have employed the binomial approximation.

Levy, Martens, and Benenson (2012) simulates a situation in which drivers search for parking in a residential neighborhood on their return from work, and in which therefore the occupancy rate increases as the evening proceeds. They compare the average realized number of parking spaces searched in their simulation model, PARKAGENT, as a function of the realized occupancy rate, to the expected number of parking spaces searched under the binomial approximation, as a function of the expected occupancy rate. When the realized occupancy rate in their simulation model is above 85%, the simulated average number of parking spaces searched is considerably higher than the expected number under the binomial approximation with that occupancy rate. Though their analysis is not steady state, and though their conclusions rest on the soundness of their simulation model, the discrepancy between their simulated numbers and those obtained under the binomial approximation is sufficiently large to cast doubt on the accuracy of the binomial approximation.

There are further reasons to doubt the accuracy of the binomial approximation. The following four apply even if entry to the parking search area is indeed generated by a time- and space-independent Poisson process.

1. The binomial approximation takes the occupancy rate over the parking area as being constant over time. But with a finite parking area, which we assume and is realistic, stochasticity results in fluctuations in the realized occupancy rate. We shall show that taking this into account results in an expected cruising-for parking time that exceeds that obtained under the binomial approximation⁶.
2. The binomial approximation is based on the assumption that the occupancy probabilities of adjacent parking spaces are statistically independent. But since parking spaces are spatially ordered, the probability that a particular parking space is occupied is higher if its upstream neighbor is occupied⁷. This positive spatial autocorrelation leads to more concentrated bunching of occupied parking spaces than would occur under the binomial approximation.
3. The binomial approximation does not account for competition between cars cruising for parking.

⁶ If a person cruising for parking experiences a realized occupancy rate of q^* throughout his search, under the binomial approximation, his expected search time *per* fn. 5 is $1/(1 - q^*)$. Thus, if the realized occupancy rate changes slowly relative to search time, under the binomial assumption the average expected search time equals the reciprocal of the harmonic mean of the realized vacancy rate, which, *per* Jensen's Inequality, exceeds the reciprocal of the mean vacancy rate.

⁷ If a parking space is vacant, then the probability that it is occupied during time unit t equals the probability that a car enters the track adjacent to the parking space during time unit t , plus the probability that a car entered the track adjacent to the immediately upstream parking space during the time unit $t - 1$ and found it occupied, plus the probability that a car entered the track adjacent to the next upstream unit in time unit $t - 2$ and found it occupied at time unit $t - 2$ and then found the parking space downstream from it occupied at time unit $t - 1$, and so on.

4. Since the parking area is finite, a car may have to circle the block to find a vacant parking space. If it does so, then the probability of its finding a vacant parking space on its second circuit is lower than the unconditional probability of its finding a vacant parking space on its first circuit.

There are also good, practical reasons to doubt that entry into a parking search area is well described by a time- and space-independent Poisson process.

5. The demand for parking is derived from the demand for activities, which are not uniformly distributed over time and space. Some locations are busier than others, and any location is busier at some points of the day than at others. Holding constant the mean vacancy rate, systematic spatial and temporal variation in the demand for parking and hence in the vacancy rate therefore increases expected cruising-for-parking time. Put alternatively, the mean vacancy rate experienced by someone cruising for parking is higher than that measured by an external observer⁸.

6. The expected cruising-for-parking times generated by the binomial approximation square with neither experience nor policy discussion. In particular, expected cruising-for-parking times calculated according to the binomial assumption seem consistently too low. Experience suggests that in a section of town where the average curbside occupancy rate is, say, 80%, finding a vacant curbside parking space reasonably close to one's destination may quite frequently be difficult. In contrast, under the binomial approximation, the expected number of parking spaces searched before finding a vacant

⁸ The demand for parking at a sports arena is an extreme example. The typical user experiences a crowded parking lot, even though the vacancy rate averaged over the day is high.

space (including the vacant space) is only five. Policy discussion in Shoup⁹ (2005) and elsewhere indicates that cruising for parking is perceived to be a serious problem in downtown areas, and yet the expected cruising-for-parking times under the binomial assumption are modest, except as the occupancy rate approaches one.

In this paper, we consider a stylized model that abstracts from the spatial and temporal inhomogenities discussed in 5. above. Space is the circumference of a circle of finite length, which we refer to as "the track". Parking spaces are evenly spaced points around the track. The arrival of drivers is generated by a time-independent and space-independent Poisson process, and where it enters is uniformly distributed around the track. Each driver cruises clockwise around the track at an exogenous speed and takes the first vacant parking space she encounters, parks there for a period of time that is determined by a draw from a time- and space-independent probability distribution (which may or may not be negative exponential), and then exits the system. The expected curbside parking occupancy rate is calculated as the expected total time that cars are parked around the track per unit time divided by the maximum total time that cars can be parked around the track per unit time. The expected total time that cars are parked per unit time equals the Poisson arrival rate of drivers times the expected parking duration, and the maximum total time per unit time simply equals the number of parking spaces

⁹ Table 11-5 in Shoup (2005) reports that average cruising-for-parking time over the 16 studies of cruising for parking that he located was 8.1 minutes, about 500 seconds. Our calibration of the numerical simulations, presented below, implies that it takes 1.800 seconds to travel from one parking space to the next. Applying this figure implies that the expected number of parking spaces searched is 278. Under the binomial approximation, this corresponds to an occupancy rate of 99.64%.

round the track. The natural measure of a unit of time in the model, which we adopt, is the time it takes to travel from one parking space to the next.

While our simulation model operates in continuous time and continuous space, Figure 1 portrays a modified version of it in which both time and space are discretized. The Figure shows a sample state of the model. The upper row of boxes represents sections of the track on which cars travel, searching for a parking space. The lower row of vertically aligned boxes represents curbside parking spaces. Cars travel from left to right, one parking space per time step. Since the track is in fact circular, the rightmost box is joined to the leftmost box. A road box is white if there are no cars on the corresponding track section at time step t . A road box is yellow if there are one or more cars on that track section at time step t , with the number in the box indicating the number of cars (which we term a cohort) on that track section. A curbside parking box is white if the curbside parking space there is vacant/unoccupied and is blue if it is occupied. Later, in Figure 3, we shall display the same diagram over successive time steps to illustrate how the system might evolve over time.

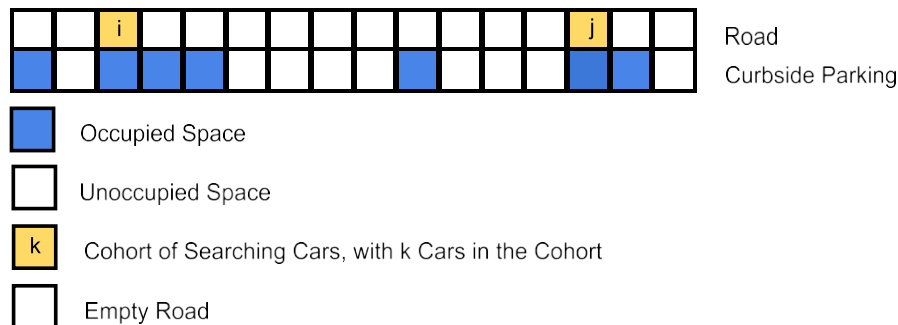


Figure 1: A Discrete Representation of a Sample State of the Simulation Model

Our model incorporates three important simplifying assumptions. First, it is spatially symmetric (except that parking spaces are points). Second, its equilibrium describes a (stochastic) steady state with exogenous, time- and space-independent Poisson (and therefore history-independent) processes generating entry to the track and exit from a parking space, conditional on entry to it. Third, as modeled, a driver's parking search "strategy" is trivial; she starts searching for parking as soon as she enters the track, and keeps on driving in the same direction until she encounters a vacant parking space, where she parks. In reality, even one-dimensional parking search is more complicated than this. With an exact destination, a rational driver does not start cruising for parking until a certain distance from her destination (Arnott and Rowse, 1999; Arnott, 2014), and if parking search is not unidirectional, a driver may decide to turn around and backtrack. Parking search strategy in two dimensions is considerably more complex.

We chose our assumptions to achieve a balance between comprehensibility, accuracy, and realism. We could have made the model even simpler. First, we could have described space more simply as a spatially ordered set of discrete parking spaces and time as discrete, as is done in Figure 1, in which case cruising for parking would be modeled as a multi-server queuing system, with unserved users moving from one server to the next between time periods¹⁰. We decided not to do so since the discretization of time and

¹⁰ Indeed, this is how Arnott and Qiong, and Arnott and Qu modeled the problem in their work.

space causes artificial difficulties in the between period modeling of events¹¹. Second, we could have ignored that servers are spatially ordered, or could even have collapsed the multi-server queue into a single-server queue, but both kinds of simplification might affect the qualitative properties of equilibrium.

The central issue that the paper addresses is whether, in stochastic steady state, the interaction between the time- and space-independent Poisson entry process, the time- and space-independent process of exit from a parking space, conditional on entry to it, and cruising for parking generates a time- and space-independent vacancy generation process. If it does, the binomial approximation is sound. If it does not, then a more sophisticated representation of the vacancy generation process is needed.

We originally explored exact analytical solution of the model, but had no substantive success. In the next section, we cast our model in the contexts of queuing theory and Markov chains, and then explain the difficulties in exact analytical solution. We then had the choice between investigating analytical solution under simplifying assumptions or proceeding to simulation. We decided to employ simulation since, without an analytical

¹¹ For example, the modeler needs to make the choice as to which happens first between time periods (between the "current" period and the "next" period), the vacation of parking spaces that occurred during the current period or the assignment of drivers who were waiting in a queue at the beginning of the current period. The modeler also needs to make the choice as to whether entrants to the system between periods are treated in the same way as drivers who were waiting in a queue at the beginning of the period. Since these sequencing decisions are not present in a continuous time model, they are an artifact of discretization. They may affect the qualitative properties of equilibrium, and even if they do not they invite confusion.

solution of the proper model, we would have no way of judging how accurate were the analytical solutions under the approximations.

Our basic finding is that the binomial approximation is a bad one. The base case parameter values are recorded in Table 1 below. There are 100 parking spaces around the track, with each parking space being represented as a point. The Poisson entry rate of cars to the track is $1/30$ per time unit, the distribution of parking times (stay lengths) is negative exponential with mean 2000 (or put alternatively the Poisson exit rate from an occupied parking space is $1/2000$ per time unit). The entry point of a car is random, and uniformly distributed around the track. The expected total parking duration per unit of time equals the entry rate times expected parking duration, which equals 66.67. The maximum parking duration per unit of time is simply equal to the number of parking spaces, 100. Thus, the expected occupancy rate is $2/3$. A unit of time is the period it takes to travel from one parking space to the next. Accordingly, the expected time it takes for an entering car to reach the first parking space is 0.5000 time units, the second parking space is 1.500, ----. Hence, the expected cruising-for-parking time equals the expected number of curbside parking spaces searched (including the last, successful search) minus 0.500. Under the binomial approximation, with an occupancy rate of $2/3$, the expected number of curbside parking spaces searched (including the last, successful search) is 3.000 (see fn. 4), which corresponds to an expected cruising-for-parking time of 2.500. In contrast, for the central base case simulation, which is the focus of section 4, the simulated mean cruising-for-parking time is 4.164, so that the ratio of the simulated mean cruising-for-parking time to the expected value obtained under the binomial

approximation is 1.666. The variance of the cruising-for-parking time under the binomial approximation is 6.000 (see fn. 4) and in the simulation is 33.32, for a ratio of 5.553.

Number of parking spaces	P	100
Distance between parking spaces		1 (normalized distance unit)
Travel time between parking spaces		1 (normalized time unit)
Poisson entry rate to track	μ	1/30 (per normalized time unit)
Poisson exit rate from occupied parking space	λ	1/2000 (per normalized time unit)
(Implied) expected occupancy rate	q	2/3

Table 1: Base Case Parameter Values

Notes: We have taken a normalized distance unit to be 1/250 ml (21.12 ft), and a normalized time unit to be 1/2000 hr (1.800 seconds)

As the expected occupancy rate increases (generated by a proportional increase in the entry rate, all other parameters being held constant), so too does the ratio of the simulated mean cruising-for-parking time to the expected value obtained under the binomial approximation. With an expected occupancy rate of 11/12, the expected cruising-for-parking time under the binomial assumption is 11.50 and the corresponding simulated mean is 100.6, giving a ratio of 8.753. The corresponding variances are 132.0 and 115800, for a ratio of 877.3. *Thus, the binomial approximation gets worse as the expected occupancy rate increases.*

Since we were unsuccessful in obtaining analytical results, we can only go so far in explaining why the probability distribution of simulated cruising-for-parking time differs from the distribution obtained under the binomial approximation. Consequently, the paper focuses on describing the simulation results and the ways in which the simulated

results differ from those obtained under the binomial approximation, from a variety of statistical perspectives.

The computer code was written so that all but two of the simulations are reproducible. This was achieved by recording the realizations of all the random variables used to generate each of the reproducible runs. Making runs reproducible permits other researchers not only to check our results, but also to supplement them by applying other statistical tools to the same data that we generated.

Section 2 places the model in the context of queuing and Markov process theory. Section 3 presents the simulation algorithm. Section 4 records the quantitative results of the central base-case simulation and compares them to the results obtained under the binomial approximation, using a variety of statistical approaches. Section 5 undertakes a variety of comparative stochastic steady-state exercises, examining how the simulated probability distributions of search times change with changes in exogenous parameters. Section 6 discusses directions for future research, and presents some concluding remarks, including comments on the policy insights from the research.

2. *Placing the Model in the Context of Queuing and Markov Process Theory*

This section draws heavily on Gross, Shortle, Thompson, and Harris (2008).

2.1 *Queuing Theory*

One might think that an approximation of our model that discretizes location, in which each parking space is viewed as a server that is occupied if the space is occupied and free if the parking space is vacant, and in which queues may form at each server, would be analytically solvable using queuing theory.

There is a standard notation used in the queuing theory literature. A queuing process is described by a series of symbols and slashes, such as $A/B/X/Y/Z$, where A indicates the arrival-time distribution, B the probability distribution of service time, X the number of parallel service channels, Y the restriction of system capacity, and Z the queue discipline. In our parking model: i) since arrivals at the track are generated by a time-independent Poisson process, the arrival-time distribution is negative exponential with the exponent equal to the Poisson arrival rate, so that, according to queuing theory notation, $A = M$ (for Markovian); ii) since each parking space is a separate service channel, $X = P$, where P is the number of parking spaces round the track; iii) in the base case, the probability distribution of service time is negative exponential with mean equal to the expected parking stay time or duration, so that $B = M$; iv) since there is no restriction on capacity, $Y = \infty$; and v) the queue discipline is not a conventional one¹², so that we set $Z = ?$. Thus,

¹² Among cars that are currently between a particular parking space and its clockwise neighbor, the car that is closest to the clockwise neighbor parking space will have priority in parking; the queue discipline is FCFS (first come, first served) in this respect. However, if a car enters the system between that parking space and its clockwise neighbor, its queuing priority is determined by the location where it enters relative to the location of other cars in the clockwise neighbor's queue; since the location of the entering car is random, the queue discipline is RSS (random selection service).

Furthermore, in this discretized version of our model, servers are spatially ordered, with unserved cars in a queue at a server at time t being in the queue at the clockwise neighboring server at time $t + 1$.

the parking model is similar to a $M/M/P/\infty/?$ queuing process. In the standard multi-server queuing process, the servers are assumed to be parallel, by which is meant there is a single queue for all the servers together. However, the parking model here has a different multi-server queuing process¹³. Thus, though our parking model can be viewed as a queuing model, it is not a standard queuing model, and indeed we have found no queuing model in the literature that describes it.

2.2 *Markov Process Theory*

A Markov process is a memoryless stochastic process, in the sense that the stochastic evolution of the system after time t is completely determined by the state of the system at time t . Markov processes are classified according to the index set of the process (whether time is discrete or continuous) and the nature of the state space of the process (whether there is a finite or infinite number of states of the system).

When the parking stay time is a negative exponential distribution, our parking model describes a continuous-time (and continuous-space), infinite Markov process. Since there are no absorbing states, we conjecture but have not proved that our model has a limiting distribution¹⁴ (hence, the model has a stationary distribution and the Markov process is

¹³ From the perspective of a driver cruising for parking, the servers are moving in a counter-clockwise direction and the driver is served by the first vacant server that passes him by. From the perspective of a server, the cars cruising for parking are moving in a clockwise direction. If the server is full, the cars cruising for parking just pass on by, while, if the server is vacant, the first car to reach the server takes the vacant parking space.

¹⁴ Let s index the possible states. Starting with state s_0 , let $p_{s,s_0}(t)$ denote the probability that the system will be in state s at time t . If in the limit as time approaches infinity,

ergodic). We are interested in the distribution of curbside parking search times associated with the limiting distribution of states of the system. Since our simulations are finite, we do not observe the limiting distribution but rather a sample of the probability distribution of states conditional on the starting state.

The dimension of the state space in our model is very high. When the parking stay time is a negative exponential distribution, as it is in our base case, the state of the system is described by the occupied/vacant status of the spatially ordered parking spaces (which entails 2^P permutations) and the positions of all the cars that are cruising for parking (which may be infinite), each of which is described over the continuous space of the track.

We could reduce the number of states of the system by approximating our model through the use of discrete time or discrete space or both. Suppose that both discretizations are employed¹⁵. A state is indexed by the spatially ordered parking spaces, and then, for each parking space, by its occupancy status and the number of cars queued at it. The state transition from one time step to the next would then be determined by the number of entries at each parking space, change in the occupancy status of each parking space, and the movement of cars queued at each space to its rightmost neighbor (representing the cruising of cars). The limiting distribution of the probabilities of the various states and of

these probabilities are independent of the initial state of the system and characterize the limiting distribution of the system.

¹⁵ This discretized model is the same as a variant of the queuing model described in the previous subsection with a negative exponential distribution of service times ($B = M$).

the cruising-for-parking search times would then be calculated from the corresponding transition matrix. We have not done this in the paper since the discretization applies approximations that might alter the qualitative properties of the limiting distributions.

The following cartoons give a rough illustration of alternative ways in which the approximation of our model that discretizes time and space can evolve over time and space. We collapse time into time steps and space into a discrete set of parking spaces, allowing for a group of cars, which we term a cohort, to be at a particular parking space.

Figure 2A shows how the cohorts of cars on the left transition to the next time period if no cars either enter the track or exit from parking. All except one car advance to the right by one parking space. The exception is one of the cohort of cars at space j at time step $t - 1$. Since parking space j is vacant at time step $t - 1$, that car occupies the

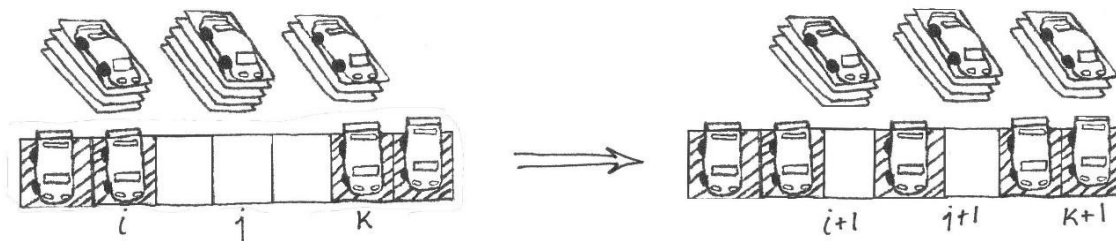


Figure 2A Cartoon of Transition from Time Step $t - 1$ to Time Step t , with no Entries and no Exits from Parking

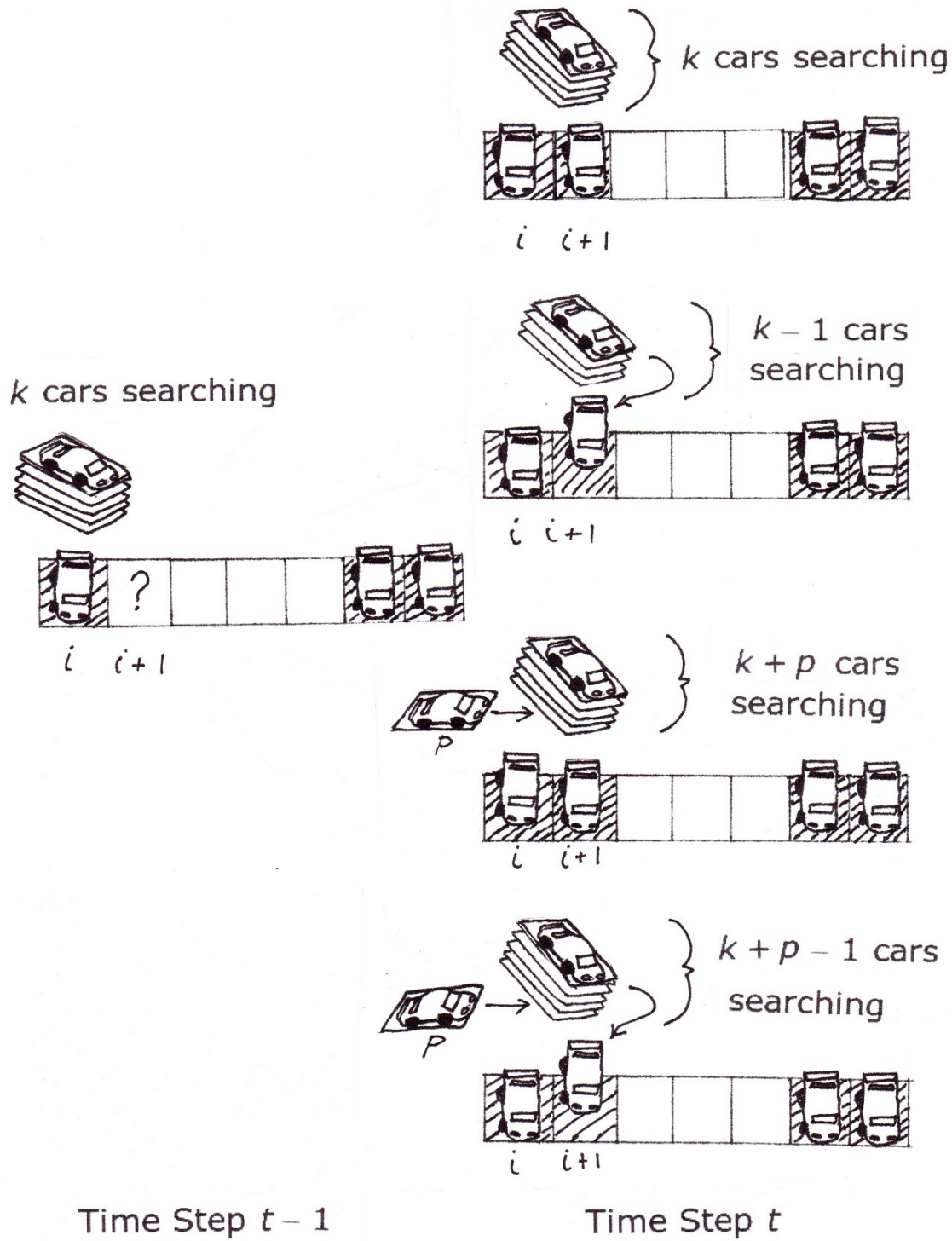


Figure 2B Cartoon of the Transitions from Time Step $t-1$ to Time Step t with No Exits from Parking

parking space, so that at time step t , parking space j is occupied and the number of cars in the cohort at parking space $j + 1$ at time step t is one less than the number of cars in the cohort at parking space j at time step $t - 1$.

Figure 2B goes one step further and shows how the state of the system at time step $t - 1$, which is displayed on the left, might transition between time steps $t - 1$ and t , when no occupied parking spaces are vacated between the two time steps and when entry to the track may occur at parking space $i + 1$. The cartoons on the right correspond to alternative states at time step t . The "?" in space $i + 1$ at time step $t - 1$ indicates that two states of the system at time $t - 1$ are considered, one in which space $i + 1$ is occupied at time step $t - 1$ (the first and third cartoons on the right), and the other in which space $i + 1$ is vacant at time step $t - 1$ (the second and fourth cartoons on the right). In the first cartoon on the right, parking space $i + 1$ is occupied at time step t and there are no entries at that space; since space $i + 1$ was occupied at time step $t - 1$, all k cars that were at occupied parking space i at time step $t - 1$ are at occupied space $i + 1$ at time step t . In the second cartoon, parking space $i + 1$ is occupied at time step t and there are no entries at that space; since space $i + 1$ was vacant at time step $t - 1$, one of the k cars that was at occupied parking space i at time step $t - 1$ takes vacant space $i + 1$ at time step t , with the other $k - 1$ cars continuing to cruise for parking. The third cartoon is like the first cartoon, except that p cars enter at occupied space $i + 1$, with the result that the cohort at that space increases from k to $k + p$ and all cars in the cohort continue cruising for parking. The fourth cartoon differs from the second cartoon in the same way that the third cartoon differs from the first.

Figure 3 depicts the variant of the model with discretized time and space as it was displayed in Figure 1, showing a sample set of transitions from time step 1 through time step 4, including entries to the track and exits from parking. At time 1, there are 7 occupied spaces on the track, which has 16 spaces, with two searching cars at occupied space 3, and three cars at occupied space 14. Between time steps 1 and 2, all the searching cars advance clockwise one parking space without encountering a vacant space, and there are no entries and exits. Between time steps 2 and 3, one car enters at space 3, two cars advance from space 4 to 5, which is occupied, a car exits parking at space 10, and three cars advance from space 15 to 16, with one parking in space 16 since it was vacant at the beginning of the time step. Between times 3 and 4, one car advances to space 4, which is occupied, one of the cars that advances from space 5 to 6 parks in space 6 since it was vacant, a car exits parking at space 14, and two cars advance from space 16 to space 1, which is occupied.

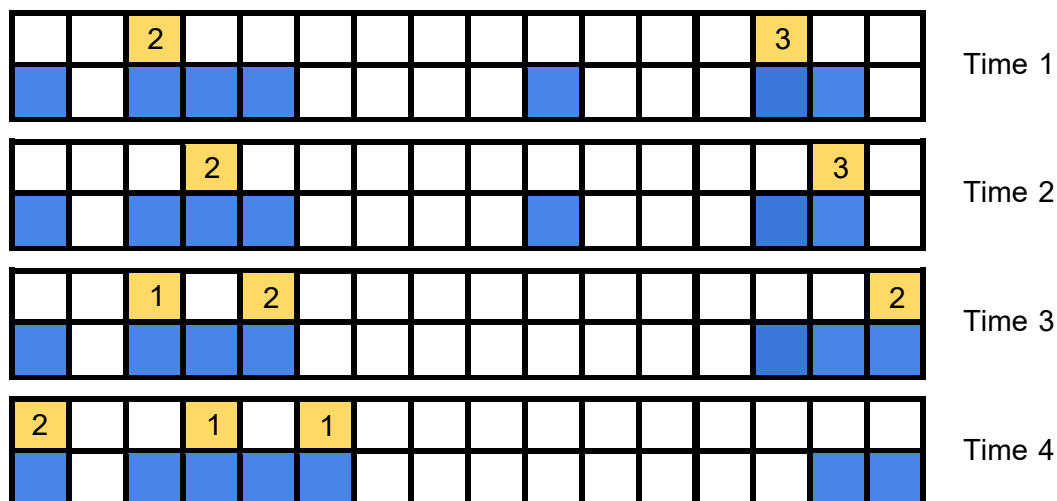


Figure 3: Depiction of the Evolution of the State of the Simulation Model
Notes: The space is in fact circular, so that a car that exits on the right simultaneously enters on the left.

The number in a yellow box is the number of cars at the space during the time unit, and is termed a cohort.

3. *The Simulation Algorithm*

Having treated how a discretized version of the model advances over time makes it easier to explain how the algorithm to solve the continuous version of the model, the one that is simulated, works.

Some details of the algorithm are chosen to either speed up computation or to ease data storage requirements. The python code for the simulation model is presented in Appendix 1, along with a hyperlink to the source code, which is also available at <http://math.ucr.edu/~parker/CruisingForParking/>. All calculations are done at 32 bit floating point precision. Here we describe the pseudo-code or program logic.

Each simulation starts by generating a pattern of occupied and vacant parking spaces around the track consistent with the binomial approximation¹⁶, and zero cars cruising for parking¹⁷. The algorithm proceeds one time unit at a time, starting at $t = 1$. Within each time unit, the algorithm has three stages: exogenous evolution, unconflicted endogenous

¹⁶ In particular, the parking status (vacant vs occupied) of each space is generated by an independent draw from a binomial distribution, with the remaining parking duration of each occupied space then generated by independent draws from the negative exponential distribution of parking durations.

The cruising-for-parking times of those cars parked at $t = 0$ do not enter the calculation of mean cruising-for-parking time.

¹⁷ Strictly this is not consistent with the binomial approximation. However, later we shall derive that the expected number of cars cruising for parking under the binomial approximation is $\mu/(1 - q) + 0.5\mu$, which with the base case parameters is 0.1167. Thus, for this and indeed all the other simulations, this assumption is innocuous.

evolution, and resolution of conflict, where a conflict is a situation in which more than one car that is cruising for parking passes the same vacant parking space.

1. Randomly generate¹⁸ the cars that enter between $t - 1$ and t , identifying each car by its entry order, and recording its entry time and location. Recall which parking spaces were unoccupied at time $t - 1$, which parking spaces remain occupied throughout the time unit, and the "vacation" time of each of occupied parking spaces that is vacated during the period. Advance all cars that were cruising for parking at time $t - 1$ one space.
2. For each of the cars that were cruising for parking at time $t - 1$, determine whether the single parking space it passed by during the time unit was occupied when it passed by. If the space was occupied, eliminate that car from the set of cars that are candidates for entry into a vacant parking space in that time unit. For each of the cars that entered the track during the time unit, determine whether it passed by a parking space during that time unit, and if it did whether the parking space was occupied at the time the car passed by it. Eliminate that car from the set of cars that are candidates for entry into a vacant parking if either it did not pass a parking space during the period or it did and the space was occupied. At this stage, one has a set of "candidate cars" that have passed this stage of the qualification to be matched with a vacant parking space. A "conflict" is a situation where more than one car is a candidate for the same vacant parking space. Assign each of the candidate cars that is "unconflicted" to the vacant parking space it passed by during

¹⁸ In the random simulation runs, the random generation is done separately for each run. In the reproducible runs, the random generation is done only for the base case run, with the realization of the randomly generated numbers being stored for subsequent runs with the same parameter values.

the time unit, recording its entry time into curbside parking and calculating its cruising-for-parking time and the number of parking spaces it searched.

3. Resolve each conflict. Among those cars that are candidates for the same vacant parking space, assign the parking space to that candidate car that first passed the parking space during the time period. Record the car's entry time into curbside parking and calculate its cruising-for-parking time and the number of parking spaces it searched.
4. Randomly generate the parking duration of each of the spaces that was occupied during the period, and, adding this to the time at which it was occupied, calculate the time at which it will be vacated. Record other relevant statistics for the time unit, such as the realized occupancy rate and number of cars cruising for parking at the beginning of the time unit, and update the occupancy status of each parking space.
5. Proceed one time unit at a time until one million cars have been assigned a vacant parking space¹⁹.

Two types of simulation run were undertaken, which differ in the amount of information was collected during the simulation run. "Reproducible runs" record the realizations of all the random variables during the run, so that the run can be exactly reproduced.

Unreproducible or "random: runs do not record these realizations and cannot be reproduced. The computation times of random runs are considerably lower since the data storage requirements are considerably less.

¹⁹ One million cars passing through the system corresponds to about 15,000 hours of traffic. The Poisson entry rate is one car every 30 time units, which corresponds to 54 seconds, and 54 million seconds corresponds to 15,000 hours.

One million cars might seem like overkill, but, as we shall see, it was not.

Only two of the simulation exercises that are reported in the paper are random. The first undertook 1000 simulations with the base case parameters and different realizations of the random variables, each with 100,000 cars finding a vacant parking space, with the aim of estimating the variability in output across simulation runs due to different stochastic realizations. Results from this set of simulations are shown in Figure 8. The second took 100 snapshots of the occupancy status of all the parking spaces, one every 1000 time units, with the aim of generating a visualization of the temporo-spatial evolution of occupied parking spaces. The results are displayed in Figures 9 and 10.

For each simulation run, results are recorded only after 10,000 time units (5 hours) have elapsed, on the assumption that, after this period of time, the distribution of cruising-for-parking times should be little affected by the randomly generated initial condition of the track.

A directory of the simulation runs is supplied in Appendix 2.

4. The Central Base Case Simulation: Different Statistical Perspectives

The parameters for the base case are given in Table 1. The "central" base case simulation is the single, reproducible base case simulation, the results of which are discussed in detail in this section. Section 4.1 compares the distribution of cruising-for-parking times in the central base case simulation to that obtained under the binomial approximation, as well as the moments. Section 4.2 describes various effects, each of which contributes to the difference between the two distributions, and measures the importance of each, and

then considers pairwise interactions between the effects. Section 4.3 comments on disparate aspects of the results.

4.1 *Comparison of the Simulated Distribution of Cruising-for-Parking Search Times to the Distribution under the Binomial Approximation*

The parameters for the base case simulation were reported in Table 1. We start in Table 2 by comparing the moments of the distribution of cruising-for-parking times for the central base case simulation to the theoretical distribution based on the binomial approximation. All the numbers presented are in normalized time units. The simulated results are based on a simulation length of 10^6 cars, which corresponds to roughly 15,000 hours, with the results from the first five hours dropped in order to ensure that the recorded results are little affected by the randomly generated initial conditions.

	Mean	Variance	Skewness ²⁰	Fisher Kurtosis
Simulated	4.164	33.32	3.589	27.72
Binomial Approximation	2.500	6.000	2.041	6.167

Table 2: Moments of the Probability Distribution of the Cruising-for-Parking Time for the Central Base Case in Normalized Time Units

As was noted earlier, the mean from the simulations is 1.666 times as large as the mean calculated under the binomial assumption, and the variance, skewness, and Fisher kurtosis are all larger too, indicating that the simulated distribution has a fatter tail.

²⁰ There are different definitions of skewness and kurtosis. The definitions employed throughout the paper are those given in fn. 4.

We digress briefly to clarify a potential source of confusion. Thus far we have presented results in terms of the number of curbside parking spaces searched, *including* the last, successful search. With this definition of the "number of curbside parking spaces searched", under the binomial approximation the moments of the distribution of the number of curbside parking spaces searched are those given in fn. 4, including the familiar result that the mean is $1/(1 - q)$, where q is the occupancy rate. We derived earlier that with this definition, expected cruising-for-parking time is $1/(1 - q) - 0.5$. In the rest of the paper, however, we define the "number of curbside parking spaces searched" as the number of curbside parking spaces searched, *excluding* the last, successful search, which is also the number of occupied spaces searched. We do this so that our plots of the number of curbside parking spaces searched start at 0 rather than at 1. Since the number of parking spaces searched including the last, successful search is one more than the number of parking spaces search excluding the last, successful search, the mean number of parking spaces searched including the last, successful search is one more than the mean number of parking spaces excluding the last, successful search. Employing the alternative definition does not alter the distribution of curbside parking search times²¹.

We could now compare the cdf for the simulated distribution of the number of curbside parking spaces searched to that of the theoretical distribution under the binomial approximation. We have chosen not to do so since the probability mass is so

²¹ Under the alternative definition of "number of parking spaces searched", the expected cruising-for-parking time equals the number of parking spaces searched plus 0.5.

concentrated for just a few parking spaces searched. The cdf's show that the simulated distribution first-order stochastically dominates the theoretical distribution, consistent with the results in Table 2. We found a "ratio plot", which plots a ratio against the number of curbside parking spaces searched, to be a more informative way of comparing the two distributions. For n curbside parking spaces searched, the numerator in the ratio is the realized proportion of cars that encountered n occupied parking spaces before finding a vacant parking space, while the denominator is the corresponding expected proportion based on the binomial approximation. Figure 4 displays the ratio plot for the base case simulation. It shows vividly how much fatter the tail is in the simulated distribution than in the theoretical distribution. Consider for example $n = 17$. Under the binomial approximation, the probability that $n = 17$ is $(2/3)^{17}(1/3) = 3.383 \times 10^{-4}$ implying that, out of 10^6 cars, the expected number of cars to encounter 17 occupied parking spaces before finding a vacant one is 338.3. The number obtained from the simulation is 5049, which is 15 times larger than that obtained under the binomial approximation. With $n = 25$, the number obtained under the binomial approximation is 13.20 and the number obtained from the simulation is 1515, which is 114 times as large as that obtained under the binomial approximation.

Whatever the probability distribution of the number of occupied parking spaces searched, the probability that a particular entering car immediately encounters a vacant parking space equals the contemporaneous vacancy rate. Thus, averaged over cars, the simulated probability of $n = 0$ equals the mean vacancy rate. All of our simulation runs are consistent with this observation.

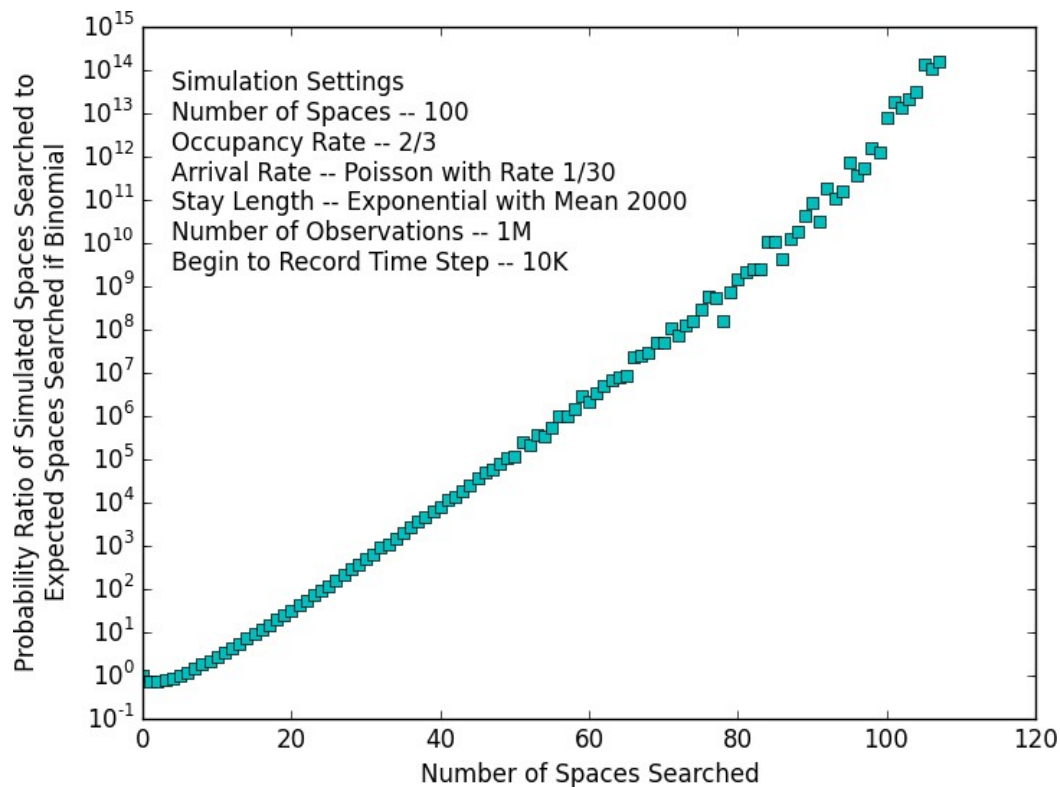


Figure 4: Ratio Plot for Occupancy Rate 2/3

4.2 Possible Effects Contributing to the Difference in the Distributions of Cruising-for-Parking Search Times

What is causing the simulated density function for cruising-for-parking time to have a tail that is so much fatter than the tail of the density function implied by the binomial approximation? Unfortunately, as explained earlier, even though our model specifies the stochastic process generating occupancies and vacancies, we have been unable to solve analytically for the implied stationary distribution of cruising-for-parking times. We can, however, identify several possible effects, each of which tells part of the story.

1. *The bunching effect*

Intuitively, the directed nature of search for a vacant parking space may lead to spatial autocorrelation, which we term the *pure bunching effect*. One measure of this effect is the expected number of occupied parking spaces encountered before a vacant space is encountered.²² To illustrate how this effect works, compare the number of occupied parking spaces searched before finding a vacant parking space under the repeated pattern VOOVOOVOO -- with that under VVOOOOVVOOOO ---, both of which have an occupancy rate of $2/3$. In the former situation, the distribution of the number of occupied parking spaces searched before finding a vacant space is 0 with probability $1/3$, 1 with probability $1/3$, and 2 with probability $1/3$, for an expected value of 1. In the latter situation, the corresponding probabilities are 0 with probability $1/3$, 1 with probability $1/6$, 2 with probability $1/6$, 3 with probability $1/6$, and 4 with probability $1/6$, for an expected value of $10/6$. Another measure of this effect is the expected number of occupied spaces in a bunch. In the former example, this number is 2; in the latter example, it is 4; and under the binomial approximation, it is 3.²³ In the central base case simulation, the mean size of a bunch of occupied spaces is 3.887 (and the variance is 19.24), which indicates that bunches are significantly more clustered in the simulation

²² Note that this is a measure of bunching from the perspective of a stationary observer, which is different from a measure from the perspective of a searching car. To illustrate the difference, suppose that bunches remain unchanged and move clockwise at the same speed as cars. Then an entering car that finds the first parking space it encounters to be occupied will continue to encounter occupied spaces.

²³ Start with an occupied space whose leftmost neighbor is vacant. The probability that there is one occupied space in the bunch is the probability that the space after the occupied space is vacant, which is $1/3$; the probability that there are two occupied spaces in the bunch is $2/3(1/3)$, ---. Thus, the expected value is $1/3 + 2(2/9) + 3(4/27) + 4(8/81)$ ---, which can be shown to equal 3.

than under the binomial approximation. We define the ratio of the mean size of a bunch in a simulation to the corresponding expected size under the binomial approximation, minus one, to be the *strength of the bunching effect*; here it is $3.887/3.000 - 1 = 0.296$. The statistic provides a measure of the degree of excess bunching.

To get an idea of the importance of the bunching effect in explaining the discrepancy between the simulated mean cruising-for-parking time and the expected time under the binomial approximation, assume for the sake of argument that the mean number of parking spaces searched (including the vacant space) equals the expected number of occupied parking spaces in a bunch, which holds under the binomial approximation. Call this "assumption A". Under assumption A, the expected cruising-for-parking time in the central base case simulation would be $3.887 - 0.5000 = 3.387$, and the bunching effect would explain a proportion $(3.387 - 2.500)/(4.164 - 2.500) = 0.5331$ of the difference between the mean cruising-for-parking time between the central base case simulation and that obtained under the binomial approximation. While the validity of assumption A is certainly open to question, this back-of-the-envelope calculation does indicate the potential importance of the bunching effect in explaining the discrepancies between the simulation results and those obtained under the binomial approximation.

Above we have documented the bunching effect for the central base case simulation, and taken a first pass at gauging its importance in explaining the discrepancy between the simulated mean cruising-for-parking time and that generated under the Poisson approximation. But we have not investigated how the bunching effect differs across

simulation runs. Nor have we developed theory that explains either how the bunching effect occurs or how it affects the discrepancy.

2. *The Jensen's Inequality effect*

Intuitively, if the track has an infinite number of parking spaces around it, by some law of large numbers, the occupancy rate should remain at $2/3$. With a finite track, however, the occupancy rate is stochastic. Suppose, for example, that the occupancy rate is $q_0 = 1/2$ half the time and $q_1 = 5/6$ the other half. If the stochastic process otherwise satisfies the binomial approximation, then the expected number of occupied parking spaces encountered before finding a vacant space is $1/2(1/(1 - q_0) - 1) + 1/2(1/(1 - q_1) - 1) = 3$, whereas it would be $1/(1 - q) - 1 = 2$ with an occupancy rate of $2/3$ all the time. This result is an application of Jensen's Inequality²⁴ since, under the binomial approximation, the function relating the expected number of occupied parking spaces searched to the occupancy rate is convex. Intuition suggests that this convexity property should hold for the simulations too. Thus, we term this effect the *Jensen's Inequality effect*.

To get an idea of the quantitative importance of the Jensen's Inequality effect, assume for the sake of argument that the expected cruising-for-parking time for a car that enters the track at a particular time equals the reciprocal on the vacancy rate at that time. Call this "assumption B", which adapts the binomial assumption to account for the stochasticity of

²⁴ In the probabilistic context, Jensen's Inequality states that if X is a random variable and f is a convex function, the expected value of f is greater than f evaluated at the expected value of X ; here X is the occupancy rate, and f is the function relating the expected number of occupied parking spaces searched to the occupancy rate.

the occupancy rate when the search area is finite. Under assumption B, the expected number of parking spaces searched before finding a vacant space (including the vacant space) would equal the reciprocal of the harmonic mean of the vacancy rate, and the expected cruising-for-parking time would equal this number minus 0.5000.

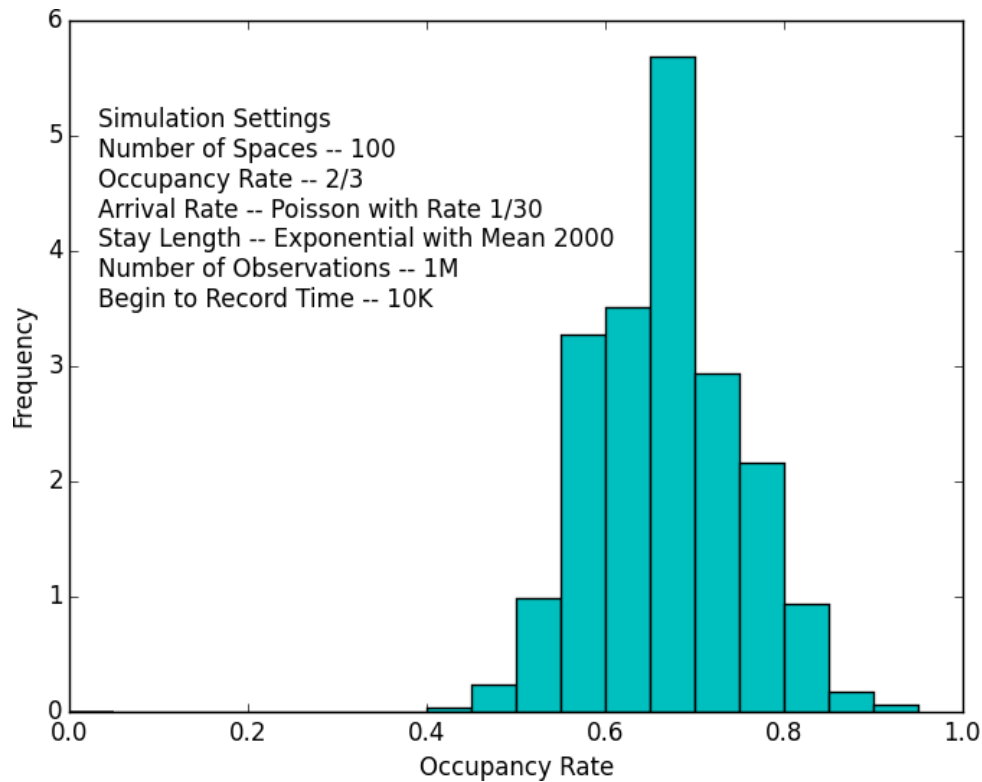


Figure 5: Frequency Distribution of the Occupancy Rate: Central Base Case

Figure 5 displays the frequency distribution of the occupancy rate for the central base case simulation. The corresponding harmonic mean of the vacancy rate is 0.2178. Define the ratio of the expected vacancy rate to the harmonic mean of the simulated vacancy rate, minus one, to be the *strength of the Jensen's Inequality effect*. Here it is $0.3333/0.2178 - 1.0 = 0.5305$.

Under assumption B, the expected number of parking spaces searched before finding a vacant parking space (including the vacant parking space) is $1/0.2179 = 4.589$, and the expected cruising-for-parking time is 4.089. Thus, under assumption B, the Jensen's Inequality effect would explain a proportion $(4.089 - 2.500)/(4.164 - 2.500) = 0.9549$ of the difference between the simulated mean cruising-for-parking time and that obtained via the binomial approximation.

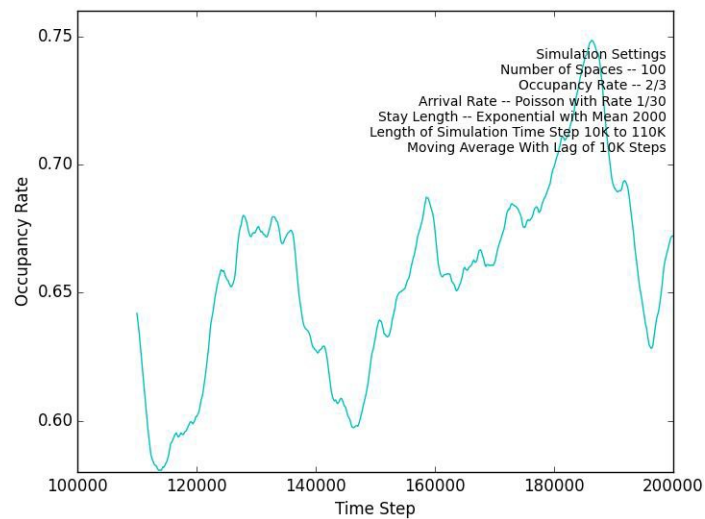
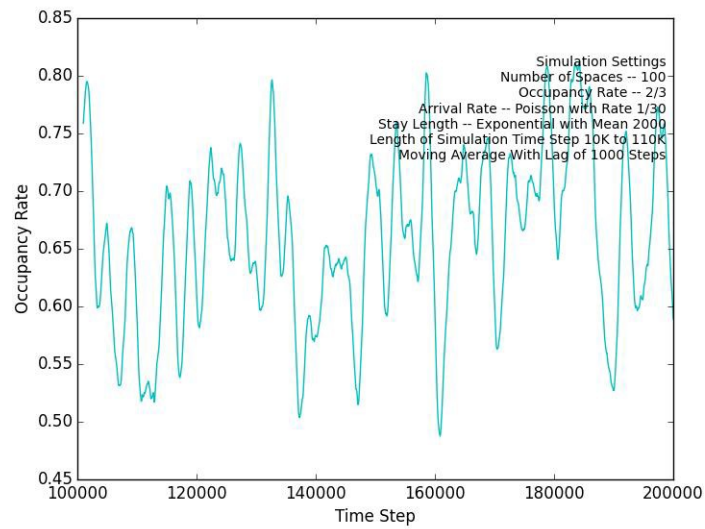
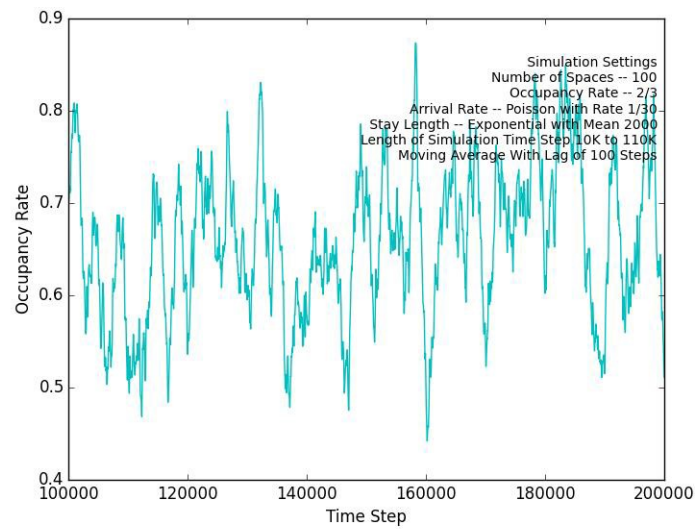
Applying assumption B likely results in overestimating the importance of the Jensen's Inequality since it assumes that each car faces an occupancy rate over its entire search equal to the occupancy rate at the time it entered the track. If, to the contrary, fluctuations in the occupancy rate are very high frequency, then each car would face an occupancy rate over its entire search equal to the expected occupancy rate, in which case the Jensen's Inequality effect would disappear. Thus, the importance of the Jensen's Inequality effect depends negatively on the frequency of fluctuations in the simulated vacancy rate.

Figure 6 gives some insight into the importance of short relative to long waves in the realized occupancy rate. Panel A plots a moving average of the occupancy rate for the central base case simulation run, where the average is over the 100 time units (3 minutes); panels B and C do the same but with an average over 1000 time units (30 minutes) and 10,000 time units (5 hours). Together the panels give an idea of the periodicity of fluctuations in the occupancy rate. Recall that the mean cruising-for-parking time for the central base case simulation run is 4.164 time units (7.495 seconds).

Over such a small time period, during which, on average, only 0.1388 cars enter the track, with, on average, the same number leaving the track, the occupancy rate will only change marginally. Since the average variation in the occupancy rate over a trip is small, in the base case simulation it appears that the Jensen's Inequality effect explains some 95% of the discrepancy between the mean cruising-for-parking time in the central base case simulation and the expected cruising-for-parking time obtained under the binomial approximation.

Figure 6: Different Moving Averages of the Occupancy Rate

Notes: Panel A: moving average over 100 time units; Panel B: moving average over 1000 time units; Panel C: moving average over 10,000 time units, each sampled every 100 time units



3. *The competition effect*

A searching car's probability of encountering a vacant parking space is negatively related to the number of cars circling the track. Consider a cruising car that is one time unit upstream from a vacant space. The larger the number of cars on the track, the higher the probability that the space will be taken by the time the car arrives at the space. We term this the *competition effect*. The binomial approximation for the expected cruising-for-parking time ignores this effect.

We now explore the distribution of the number of cars circling the track under the approximating assumption that the underlying stochastic process determining the number of cars cruising the track is a birth-and-death process with birth rate μ and death rate $n(1 - q)$, where n is now the number of cars cruising for parking. Cars enter the track at Poisson rate μ . Under the binomial approximation, each of the n cars cruising for parking finds a vacant parking space at the Poisson rate $1 - q$, and since, also under the binomial approximation, these Poisson rates are independent, the aggregate rate at which cars cruising for parking "die" (by one of the cars finding a vacant space) is $n(1 - q)$. Let $p_n(t)$ be the probability that the number of cars cruising for parking at time t is n . Then we have that

$$p_n(t + dt) = p_n(t)(1 - n(1 - q)dt - \mu dt) + p_{n-1}(t)\mu dt + p_{n+1}(t)(n + 1)(1 - q)dt \text{ for } n = 1, \dots, \infty$$

$$p_0(t + dt) = p_0(t)(1 - \mu dt) + p_1(t)(1 - q)dt \quad (1)$$

The interpretation of the second equation is as follows. During the increment of time dt , it is infinitely more likely that one state transition will occur than that more than one state transition will occur. The probability that the number of cars cruising for parking is zero

at time $t + dt$ is therefore (the probability that zero cars were cruising for parking at time t multiplied by the probability that there was no entry to the track during the increment of time, $1 - \mu dt$) plus (the probability that one car was cruising for parking at time t multiplied by the probability that the car found a vacant parking space, $(1 - q)dt$). The interpretation of the first equation is similar. Letting p_n be the steady-state probability that the number of cars cruising for parking is n , (1) implies the recursion

$$p_n = (\mu/[n(1 - q)])p_{n-1} \quad \text{for } n = 1, \dots, \infty. \quad (2)$$

From this, we obtain the infinite series

$$\sum_{n=0}^{\infty} p_n = 1 = p_0(1 + z + z^2/2! + \dots) \quad \text{where } z \equiv \mu/(1 - q). \quad (3)$$

The value of the infinite series is e^z . Thus, $p_0 = e^{-z}$, $p_1 = ze^{-z}$, $p_2 = z^2e^{-z}/2!$, etc. The mean number of cars cruising for parking is therefore

$$m = 0e^{-z} + 1(ze^{-z}) + 2(z^2e^{-z}/2!) + 3(z^3e^{-z}/3!) + \dots = z. \quad (4)$$

In the base case, $z = \mu/(1 - q) = (1/30)/(1/3) = 0.1$, which implies that $m = 0.10$, $p_0 = 0.9048$, $p_1 = 0.9048E-1$, $p_2 = 0.4524E-2$, $p_3 = 0.1508E-3$, $p_4 = 0.377E-5$, ---. The variance is

$$\text{var} = z^2e^{-z} + (1 - z)^2(ze^{-z}) + (2 - z)^2(z^2e^{-z}/2!) + \dots,$$

which after some tedious algebra can be shown to reduce to z , which equals 0.1 in the base case.

The above approximation ignores that, on average, a car travels 0.5000 distance units before encountering a parking space, taking 0.5000 time units. With a Poisson entry rate of 1/30 cars per time unit, the expected number of cars on the track that have not yet reached the first parking space is $0.5000/30 = 0.01667$. Taking this into account, the

adjusted binomial approximation of the expected number of cars cruising for parking is 0.1167.

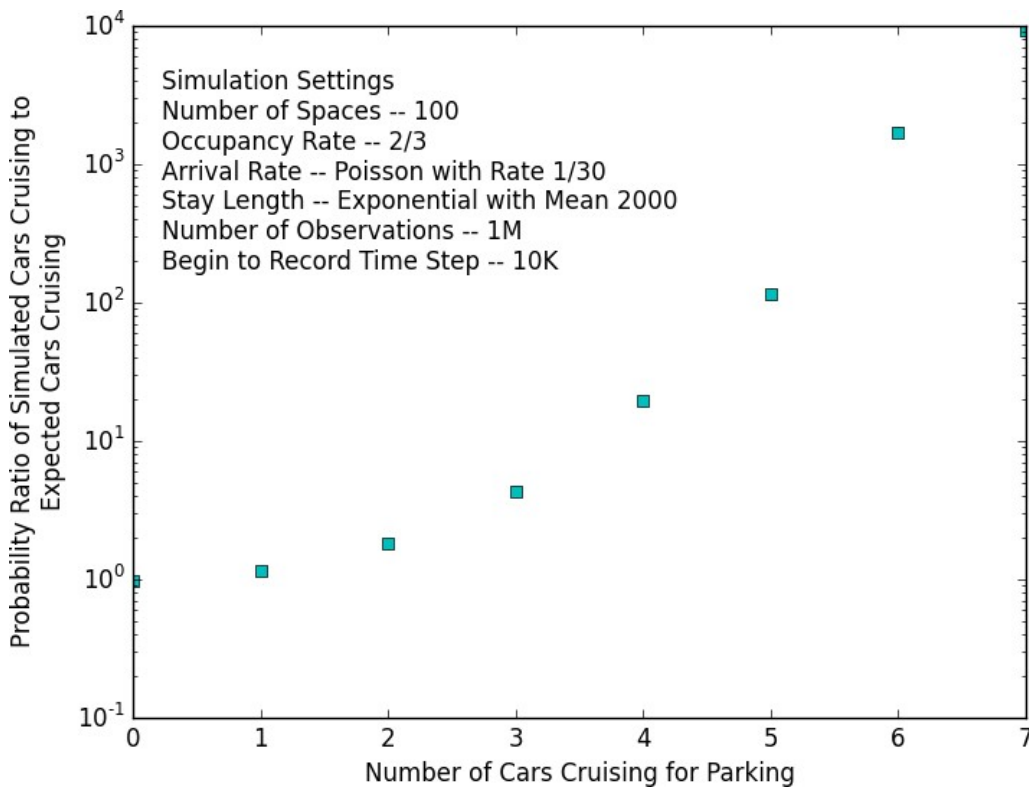


Figure 7: Ratio Plot for the Number of Cars Cruising for Parking for Occupancy Rate 2/3

Figure 7 gives the ratio plot of the number of cars cruising for parking in the central base case simulation. The corresponding mean number of cars cruising for parking is 0.1221.

Thus, at least for the central base case simulation, the birth-and-death/Poisson approximation for the expected number of cars cruising for parking is quite accurate.

Even though the probabilities of 2, 3, --- cars cruising for parking is considerably higher in the simulation than under the binomial approximation, there is so little weight in the right tail that the mean is not much higher. As we shall note later, however, this is not the

case with for the simulation run with an expected occupancy rate of $5/6$, and may not be the case for other simulation runs with an expected occupancy rate of $2/3$.

The above calculations may be employed to estimate the number of other cars on the track when a car is cruising for parking. Under the Poisson/birth-and-death approximation, this equals the expected number of cars cruising for parking conditional on at least one car cruising for parking, minus one, which can be shown to equal 0.05083. Under the binomial approximation, from the perspective of a car cruising for parking, the probability that the next parking space will be vacant is $1/3$, so that the probability that the next parking space will be vacant and that no other cruising car reaches it before the car in question is $(1/3)(1 - 0.05083/100) = 0.3332$. Under these assumptions, therefore, with an occupancy rate of $2/3$, the competition effect is quantitatively unimportant.

Applying the same logic but with the observed distribution of the number of cars cruising for parking (recorded in Table 3) in the central base simulation gives an expected number of other cars on the track when a car is cruising for parking of 0.07474, for which the competition effect is also quantitatively unimportant. Again, as we shall note later, the competition effect does become important for high occupancy rates, where the mean number of cars cruising for parking is considerably higher, and may be significant even for some simulation runs with an expected occupancy rate of $2/3$.

4. *The cruising-the-block effect*

To illustrate the next effect, suppose that there are only 10 parking spaces round the track, that a car has circled the track without finding a vacant space, and that the car is

and remains the only car cruising for parking. Under our assumed distribution of parking stay times, conditional on the car having circuited the track without having found a vacant space, the probability that the first parking space it encountered upon entering the track is still occupied on its second circuit is one minus the probability that it was vacated during the car's first circuit (that probability is $\int_0^{10} \lambda e^{-\lambda t} dt = 0.004988$, where $\lambda = 1/2000$ is the Poisson rate at which a parking space is vacated), which equals $e^{-1/200} \approx 0.9950$. The same argument applies to the second and so on occupied spaces that the car encountered upon entering the track, and also to its subsequent circuits of the track. The obvious term for this effect is the *circling-the-block effect*. With the negative exponential distribution of parking durations, the rates at which occupied parking spaces are vacated are statistically independent. The probability that a car makes a second circuit of the track without encountering a vacant parking space, contingent on its not having found a vacant space on the first circuit and its continuing to be the only car cruising for parking, is then approximately $(0.995)^{10} = 0.9512$. The probability that it makes an $n+1^{\text{st}}$ circuit of the track without encountering a vacant parking space on the n^{th} circuit is the same.

Since the probability that a car makes a full circuit of the track is higher, the higher is the expected occupancy rate and the smaller the number of parking spaces, the circling-the-block effect is more important the higher is the expected occupancy rate and the smaller the number of parking spaces.

In the central base case simulation, only 27 cars out of the one million circled the block, and only 1 of the 27 circled twice. Thus, in this simulation the cruising-the-block effect is

of negligible importance. As we shall see, however, this may not be the case for other simulations with the same parameter values. In the simulation with the same parameter values as in the base case, but with an expected occupancy rate of $5/6$ rather than $2/3$, 24,345 cars circled the block at least once, and 1 went around 41 times. In the corresponding simulation run with an expected occupancy rate of $11/12$, 177,557 cars circled the block at least once, and 2 went around over 200 times.

5. *Interaction effects*

Thus far we have identified four different effects, each of which goes part of the way towards explaining why the simulated mean cruising-for-parking time significantly exceeds that obtained under the binomial approximation: the bunching effect, the Jensen's Inequality effect, the competition effect, and the cruising-the-block effect. We considered each of these effects in isolation. But there may also be important interaction effects.

These interaction effects may cause the combination of pairs of effects on mean cruising-for-parking time to be subadditive or superadditive. There are six pairs of effects. Here we shall consider only three.

The most obvious of the pairwise interaction effects is that between the competition effect and the cruising-the-block effect. We argued above that, if there is a single car cruising for parking, the probability that it fails to find a vacant parking space on its second circuit of the track conditional on its having failed to find a vacant parking space

on its first circuit, is $(0.995)^{10} = 0.9512$, which is the probability that none of the parking spaces it passed by on its first circuit has been vacated by the time of its second circuit. Now modify the problem by introducing a second car that enters the system at the time the first car has completed its first circuit. The introduction of the second car reduces the probability that the first car finds a vacant parking space on its second circuit, conditional on having failed to find a vacant parking space on its first circuit. It will fail to find a vacant parking space on the second circuit not only if none of the parking spaces it passed by on its first circuit has been vacated by the time of its second circuit, but also if one of the parking spaces it passes by on its first circuit has been vacated by the time of its second circuit but has been taken by the second car. We term the interaction between these two effects the *multiplication* effect because, if the number of cars cruising for parking were to persist and if parking were to remain saturated, expected cruising-for-parking time would equal the expected cruising for-parking-time with one car cruising for parking multiplied by the number of cars cruising for parking.

The next pairwise interaction effect we consider is that between the Jensen's Inequality effect and the competition effect. Both intuition and the earlier discussion suggest that the number of cars cruising for parking and the realized occupancy rate are positively correlated. This is demonstrated in Table 3, which for the central base case run displays the realized probability distribution of the number of cars cruising for parking and the mean occupancy rate conditional on the number of cars cruising for parking. The positive correlation between the number of cars cruising for parking and the occupancy rate conditional on the number of cars cruising for parking fattens the right tail of the

distribution of cruising-for-parking times. For want of a better term, we term this the *correlation effect*. A statistical accident in which fewer than the expected number of cars vacate their parking spaces has a direct positive effect on the realized occupancy rate, which in turn increases the expected number of cars cruising for parking. A statistical accident in which more than the expected number of cars enters the track has a direct effect on both the realized occupancy rate and the number of cars cruising for parking. Thus, the Jensen's Inequality effect and the competition effect on the expected cruising-for-parking time are superadditive.

Number of Cars Searching, k	0	1	2	3	4	5
Probability of k Cars	0.8876	0.1033	0.008248	0.0006532	0.00007457	0.0000087
Mean Occupancy	0.6622	0.6979	0.7401	0.7923	0.8733	0.8964

Table 3: Probability of k Cars Searching and Mean Occupancy Rate with k Cars Searching with Occupancy Rate $2/3$

The final interaction effect we consider is that between the two effects that appear to be of dominant importance in the central base case, for which the expected occupancy rate is $2/3$, the bunching effect and the Jensen's Inequality effect. Earlier we defined the strength of the bunching effect to be the ratio of the simulated mean number of occupied parking spaces in a bunch to the corresponding expected number under the binomial approximation, and the strength of the Jensen's Inequality effect to be the ratio of the arithmetic mean of the vacancy rate to the harmonic mean. Under the binomial approximation, the expected size of bunches equals the reciprocal of the occupancy rate.

If this relationship were to hold true as the realized occupancy rate varies, then the strength of the bunching effect would be the same as the strength of the Jensen's Inequality effect. The intuition we earlier presented for the pure bunching effect was instead based on positive spatial autocorrelation of occupied spaces deriving from the directed nature of parking search, independent of the variability in the realized occupancy rate. Unfortunately, the excess bunching observed in the simulation confounds excess bunching due to the pure bunching effect and excess bunching deriving from variability in the realized occupancy rate. Isolating the two sources of excess bunching would require simulating cruising for parking on a very large circle, for which excess bunching deriving from variability in the realized occupancy rate would be absent.

In the base simulation, the strength of the Jensen's Inequality effect (0.530) is greater than that the strength of the bunching effect (0.296). One possible explanation is that, contrary to our intuition, the pure bunching effect is negative. An alternative explanation that we find more plausible is that the mean size of bunches demonstrates greater persistence than the realized occupancy rate.

4.3 *Comments*

1. *Differences in realized simulation base case runs*

When we started our simulations, we fixed the simulation time at 10^6 time units. But then, unexpectedly, we found that the mean cruising-for-parking time was substantially different from run to run, so switched to simulations runs in which 10^6 cars were simulated, which as noted earlier corresponds in unnormalized time units to about 15,000

hours. To investigate how different the mean cruising-for-parking time is across runs, we reran the base case simulation for 10^5 cars (not the 10^6 cars that were used in the central base case simulation) 1000 times. The top panel of Figure 8 displays the probability density function of the means across runs of the number of parking spaces searched, and the lower panel the corresponding distribution of the variances. The only explanation that we can think of for why there is so much variability from run to run, despite the large number of cars on each run, is "disasters" -- very low probability but extreme events. Even for one hundred thousand cars, there may not even be a minor disaster, but a major disaster on a run may substantially increase both the mean and the variance of cruising-for-parking time. A major disaster is "a perfect storm" that comes about purely by chance, when stochastic realizations are such that not only does parking become almost completely blocked but also it takes an exceptionally long period for parking to unblock, either because the realized entry rate remains abnormally high, or because the rate at which parking spaces are vacated remains abnormally low, or because the number of cars cruising for parking remains high. The few major disasters that occurred in this set of simulation runs can be identified as the right-tail outliers in the Figures. Thus, even though the competition effect and the cruising-the-block effect were of negligible

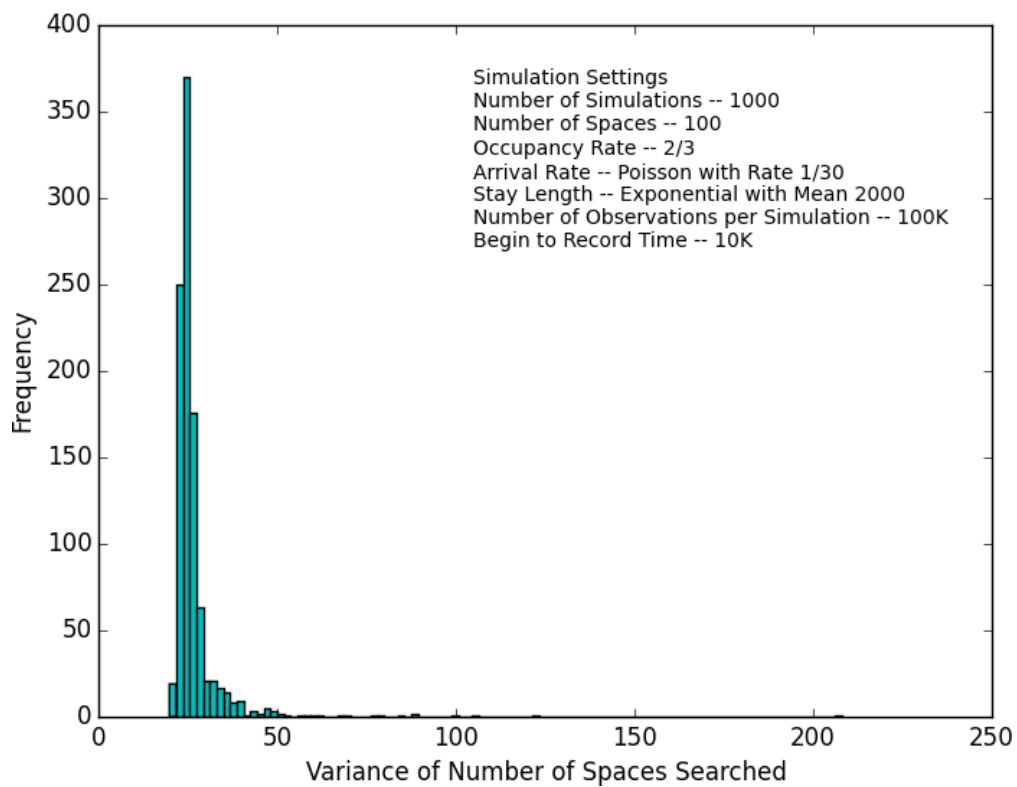
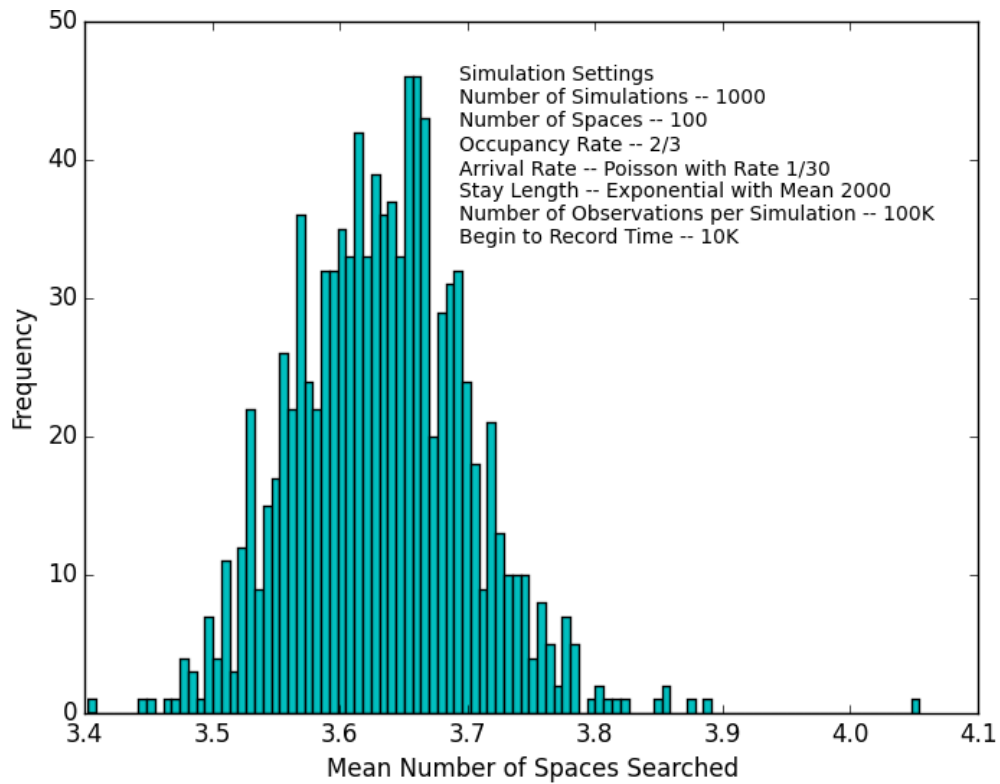


Figure 8, Panel A: Distribution of Number of Parking Spaces Searched for 1000 Simulations with Occupancy Rate = $2/3$

Panel B: Distribution of the Variance of the Number of Parking Spaces

Notes: Each simulation is for 10^5 time units.

Mean of means: 3.633. 95% confidence interval of means²⁵ (3.527, 3.745)

Mean of variances: 26.79. 95% confidence interval of variances (22.32, 36.53).

importance in the central base case simulation²⁶, the same may not be the case for other simulations with the same exogenous parameter values.

2. *Another perspective on the bunching effect*

We started with the broad intuition that, if the parking on an entire block becomes fully occupied, whether by statistical accident or by some special event, it will have ripple effects in space-time. Cars that were intending to park on that block will move to neighboring blocks in their search for curbside parking. This got us to thinking about how bunches of occupied parking spaces evolve over space-time. We started with a snapshot of the track, such as that shown in Figure 9 below.



Figure 9: Snapshot of Street with Mean Occupancy Rate $2/3$

²⁵ Computed so that 2.5% of the probability mass lies to the left of the interval and 2.5% of the mass to the right of the interval.

²⁶ The mean number of (occupied) parking spaces searched in the central base case simulation, 3.664, is very slightly above the mean of the means in top panel of Figure 8, 3.633. Thus, by happy accident, the central base case simulation is an average run.

Notes: White spaces are occupied, black spaces are vacant. There are 70 occupied spaces and 30 vacant spaces. The vacant spaces are 2, 3, 4, 6, 7, 14, 17, 21, 23, 24, 25, 26, 27, 28, 29, 31, 33, 35, 37, 38, 48, 50, 54, 60, 63, 70, 71, 73, 78, 87.

We then aimed to examine how bunches evolve -- expanding, contracting, forming, and dissolving. We had little success but did obtain one analytical results. Bunches tend to move clockwise over time since there is a higher than average probability that the first vacant space after a bunch of occupied spaces becomes occupied, and a lower than average probability that the first vacant space before a bunch of occupied spaces becomes occupied. Under the binomial approximation, the unconditional arrival rate of a car at a particular parking space is Poisson at the arrival rate at the track, $1/30$, divided by the number of parking spaces around the race track, which is 100, times the expected number of parking spaces searched before finding a vacant space (including the vacant space), which is 3.0, for a total of $1/1000$. But conditional on an initial situation where at time t there is a bunch of six occupied spaces followed by a vacant space, the rate at which cars arrive at that vacant parking space is different. The probability that a car that entered the track in the previous time unit passes that vacant space in time interval $[t, t + dt]$ is $dt/3000$; the probability that a car that entered the racetrack during the time unit before that passes that vacant spot in the same time interval is $dt/3000$ times that the probability that the left-side neighbor was occupied when the car passed it by, conditional on its being occupied at time t , which is approximately $1999/2000$; ---. Thus, the expected rate at which cars arrive at the vacant parking space is somewhat less than $7/3000$. This is the rate at which the front edge of the bunch of occupied spaces moves forward. By an

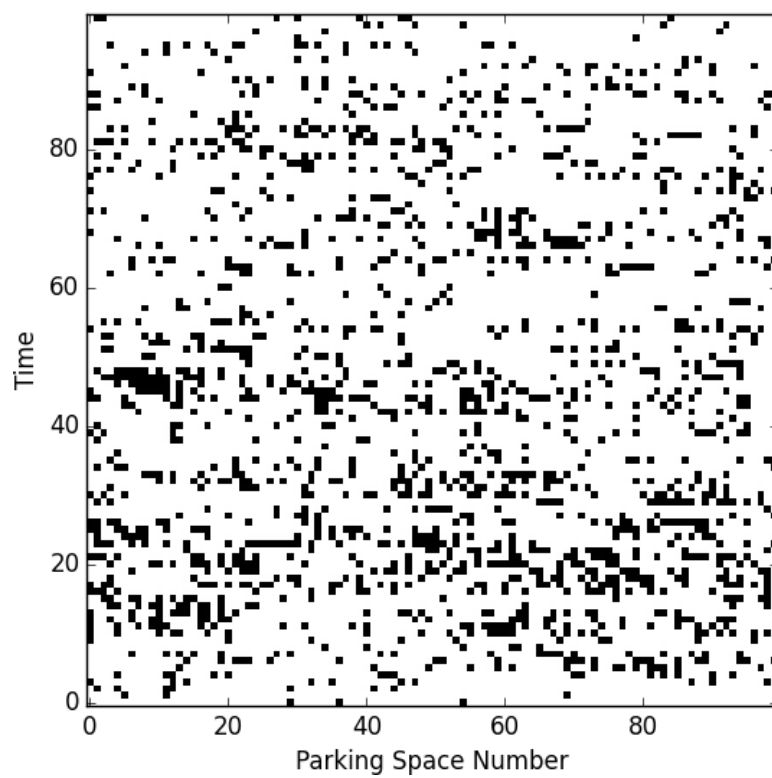
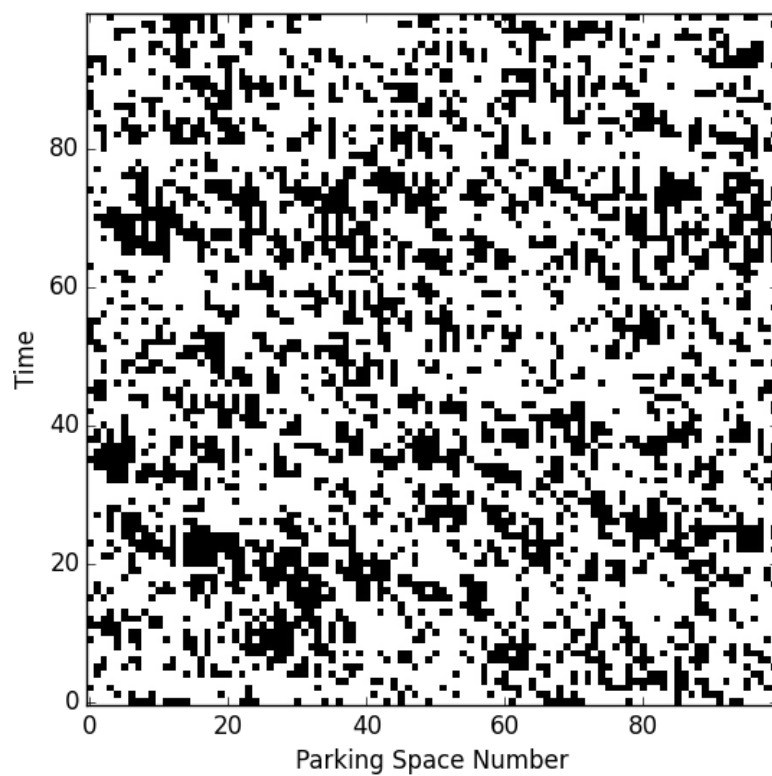


Figure 10, Top Panel: Snapshots of the Circle Every 1000 Time Steps with Occupancy Rate = $2/3$

Notes: White is occupied, black is vacant

Figure 10, Bottom Panel: Snapshots of the Circle Every 1000 Time Steps with Occupancy Rate = $5/6$

Notes: White is occupied, black is vacant

analogous argument, the probability that the first vacant space before the bunch is occupied in the same time interval is less than $1/1000$.

Unfortunately, we were unsuccessful in uncovering a body of literature on the temporo-spatial evolution of bunches. The best we could do was to take snapshots of the racetrack every so many time steps. Figure 10 displays such a snapshot, taken every 1000 time units (every half hour) with occupancy rate $2/3$ (top panel) and $5/6$ (bottom panel).

Occupied spaces are white, and vacant spaces black. Bunching shows up as positive horizontal correlation between white spaces (or black spaces); standard autocorrelation as positive vertical correlation; and the type of temporo-spatial autocorrelation discussed in the previous paragraph as positive correlation in a northeast direction. To the naked eye, bunching and standard autocorrelation are evident but not positive correlation in the northeast direction.²⁷

3. *Finiteness of the parking circle*

Some of the effects that we have identified as contributing to the discrepancy between the simulated distribution of cruising-for-parking times and the distribution obtained under the binomial approximation arise from the finiteness of the parking circle. In particular,

²⁷ While we have not investigated this, at high occupancy rates intuitively bunching should also occur among cars cruising for parking.

by some law of large numbers, on an infinite circle the mean occupancy rate would be constant over time, so that the Jensen's Inequality effect would disappear. As well, the circling-the-block effect would disappear.

On one hand, if we had known at the time we designed the simulations how much the size of the parking circle influences the outcomes, we would have undertaken some simulation runs with "large" parking circles, since this would have allowed us to separate out the effects that derive from the finiteness of the parking circle from other effects. On the other hand, small and medium-sized parking circles are more realistic since actual parkers search in the neighborhood of their destinations, which contains only a small or medium-sized number of parking spaces.

We could have undertaken considerably more statistical analysis of the central base case run, and we could have proceeded more formally. Nevertheless, we judge that the statistical evidence we have accumulated makes a compelling case that the binomial approximation is a poor one, and that the expected cruising-for-parking time calculated using it significantly underestimates the true expected cruising-for-parking time. The next section investigates the effects of parameter changes one at a time. Among other things, it will: i) show that the binomial approximation becomes increasingly poor as the occupancy rate increases; ii) provide strong evidence of the circling-the-block effect; and iii) present some evidence that the behavior of the parking system is quite sensitive to the turnover rate, holding constant the expected occupancy rate, but is insensitive to the distribution of stay times, holding constant the average stay time.

5. Comparative Stochastic Steady States

We start off by examining how the stochastic steady state changes as the Poisson entry rate changes, *ceteris paribus*. A change in the Poisson entry rate results in a proportional increase in the expected occupancy rate. Table 4 shows the moments of the distribution of cruising-for-parking times when the vacancy rate is successively halved. In the central base case, the occupancy rate is $2/3$ and the vacancy rate is $1/3$; a halving of the vacancy rate to $1/6$ results in an increase in the occupancy from $2/3$ to $5/6$; and a further halving of the vacancy rate to $1/12$ results in an increase in the occupancy rate from $5/6$ to $11/12$. For each occupancy rate, the results are for only a single simulation run of a million cars, and, as we have seen, the results for each case can be quite different across simulation runs. The number in each cell is computed from the corresponding simulation run. The number in brackets in a cell gives the corresponding number according to the binomial approximation.

Occupancy Rate	Mean	Variance	Skewness	Fisher Kurtosis
2/3 (base case)	4.164 (2.500)	33.32 (6.000)	3.589 (2.041)	27.71 (6.167)
5/6	16.70 (5.500)	3583 (30.00)	15.69 (2.640)	425.3 (6.033)
11/12	100.6 (11.50)	115800 (132.0)	12.36 (2.760)	294.5 (6.008)

Table 4: Moments of the Distribution of Cruising-for-parking Times in Normalized Time Units

Notes: For each case, the principal numbers are from a single simulation with 10^6 cars; the numbers in parenthesis are those generated by the binomial approximation.

Table 4 demonstrates that the ratio of the mean cruising-for-parking time for a simulation to the corresponding expected value obtained from the binomial approximation becomes increasingly large as the expected occupancy rate increases. In other words, the binomial approximation underestimates the mean cruising-for-parking time by an increasingly large proportion, the higher is the expected occupancy rate.

We were curious concerning the accuracy of the binomial approximation when the expected occupancy rate is low. Figure 11 displays our now-standard ratio plot when the occupancy rate is 0.1. The mean cruising-for-parking time is 0.6174. The expected cruising-for-parking time according to the binomial approximation is $1/0.9 - 0.5 = 0.6111$. Even though the binomial approximation provides an accurate approximate of the actual mean cruising-for-parking time, it still significantly underestimates the fatness of the right tail of the distribution.

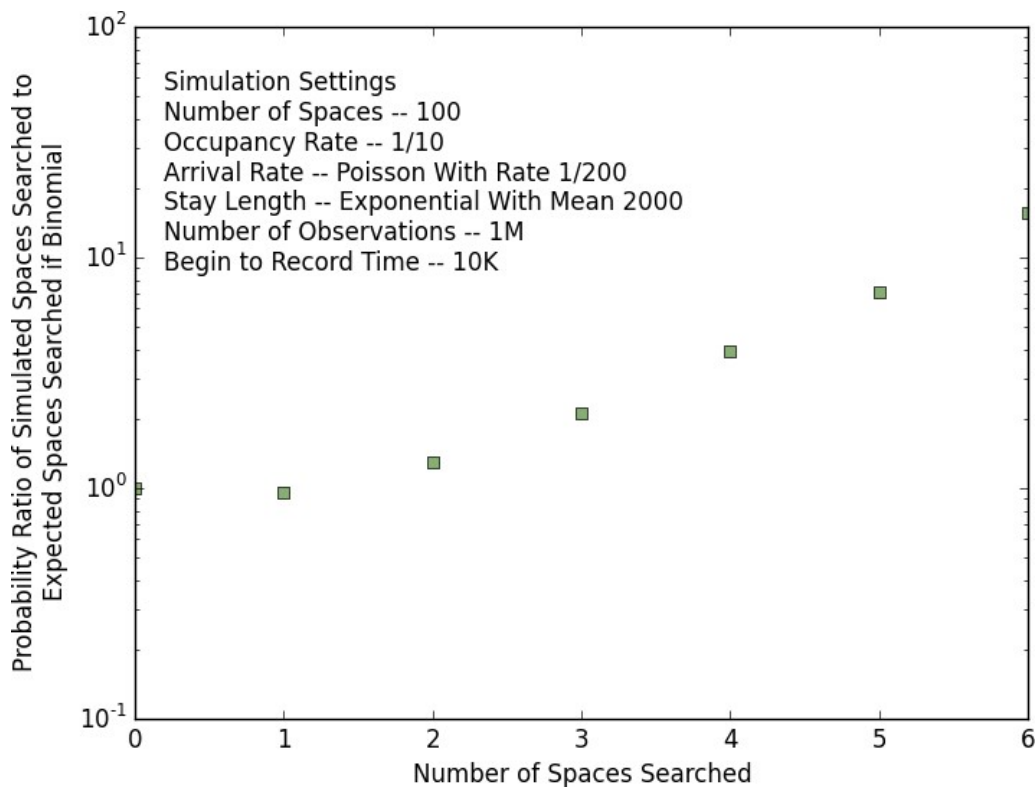


Figure 11: Ratio Plot with a Mean Occupancy Rate = 0.1

Figure 12 below presents two pairs of panels. The lower panel on the left shows a smoothed (moving average over 100 time steps or 3 minutes) time series for the number of cars on the track and the upper panel the corresponding time series for the occupancy rate, both for the case where the expected occupancy rate is $2/3$, over a period of 10,000 time steps (5 hours). The corresponding panels on the right are for the case where the expected occupancy rate is $5/6$. The results confirm the positive correlation between the number of cars searching and the occupancy rate, show the greater volatility of the number of cars searching than of the occupancy rate, and display the sensitivity of the (smoothed) number of cars searching to the occupancy rate. In the central base case simulation, the mean number of cars cruising for the parking over the entire simulation

run is 0.1221, which is close to the expected number under the binomial approximation.

When the expected occupancy rate rises to $5/6$, the mean number of cars cruising for parking over the entire simulation rises sharply to 0.6751, considerably above the expected number under the binomial approximation, which is 0.2208. Also, there are

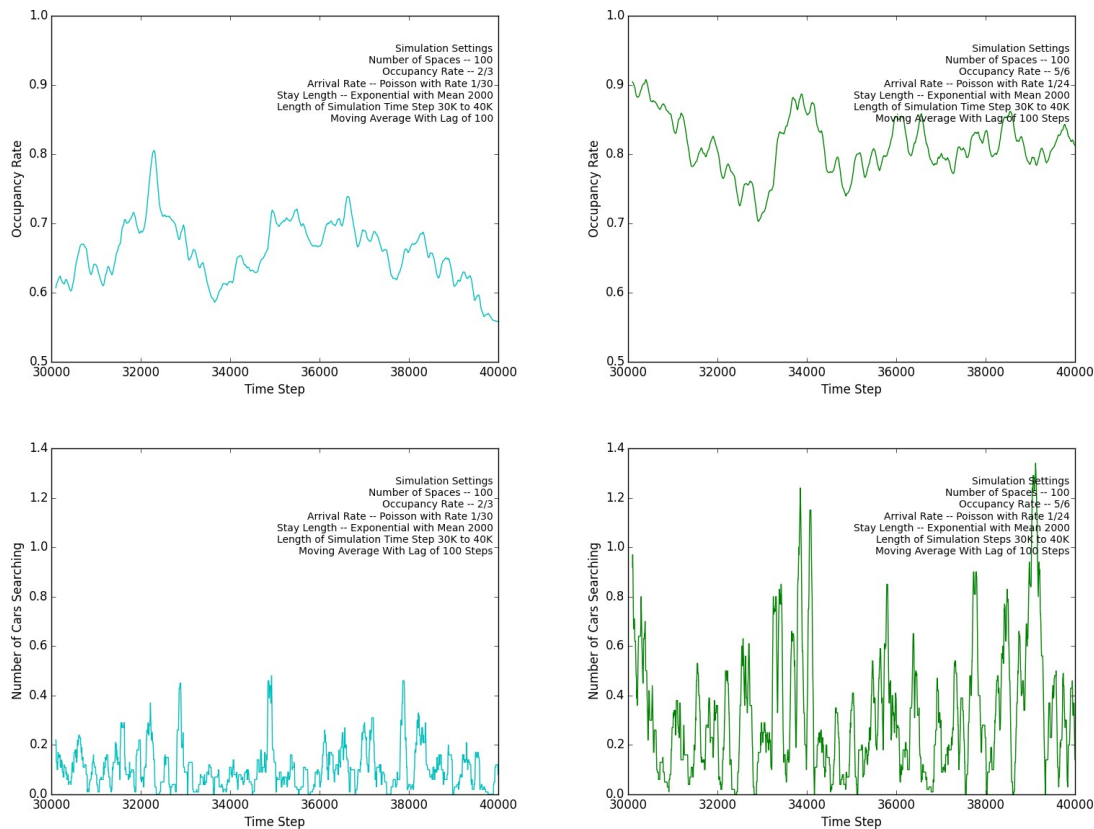


Figure 12: Smoothed Time Series for Number of Cars on the Track against the Occupancy Rate

Notes: The lower panel on the left shows the smoothed time series for the number of cars on the track, and the upper panel the left shows the time series for the occupancy rate, both with the occupancy rate = $2/3$. The panels on the right show the same with the occupancy rate = $5/6$.

are rare events in which the number of cars simultaneously on the track exceeds twenty, and at one point in time the number exceeds 45. The obvious conjecture is that these rare events occur when parking around the track becomes almost completely occupied, so that

the cars cruising round the track accumulate, with a significant number of them making more than one circuit of the track. The circling-the-block effect is magnified by the accumulation of cars searching for parking. This conjecture is consistent with the ratio plot of the distribution of number of spaces searched for the full simulation run with an occupancy rate of $5/6$, which is not shown, in which the maximum number of spaces searched is 4120.

Table 5 gives the moments for three cases that differ in the number of parking spaces around the track, with the entry rate adjusted to maintain the expected occupancy rate. The first row repeats the results for the central base case simulation in which the number of parking spaces is 100, the second row gives the results when the number of parking spaces equal 1000, and the third row the results when the number equals 10.

Parameters	Mean	Variance	Skewness	Fisher Kurtosis
Central Base Case P = 100	4.164	33.32	3.589	27.71
Base Case except P = 1000 and Arrival Rate = 1/3	3.622	23.98	3.612	22.90
Base Case except P = 10 and Arrival Rate = 1/300	108.6	190600	9.063	148.4

Table 5: Moments of the Distribution of Cruising-for-parking Times in Normalized Time Units with Occupancy Rate $2/3$

Increasing the number of parking spaces around the track by a factor of 10 relative to the base case, holding their spacing the same, has only a modest effect on the moments.

However, decreasing the number of parking spaces around the racetrack by a factor of 10 strongly increases both the mean and variance of cruising-for-parking times. The obvious conjecture is that the circling-the-block effect is important with $P = 10$ but not with $P = 100$ (as we have shown) or with $P = 1000$. The ratio plots for these cases, which are shown in Figure 13, confirm this conjecture. For $P = 10$, the circling-the-block effect would result in a sharp drop in the ratio between the last space before the block is circled, which is space 10, and the first space after the block is circled, which is space 11, and that is what is observed.

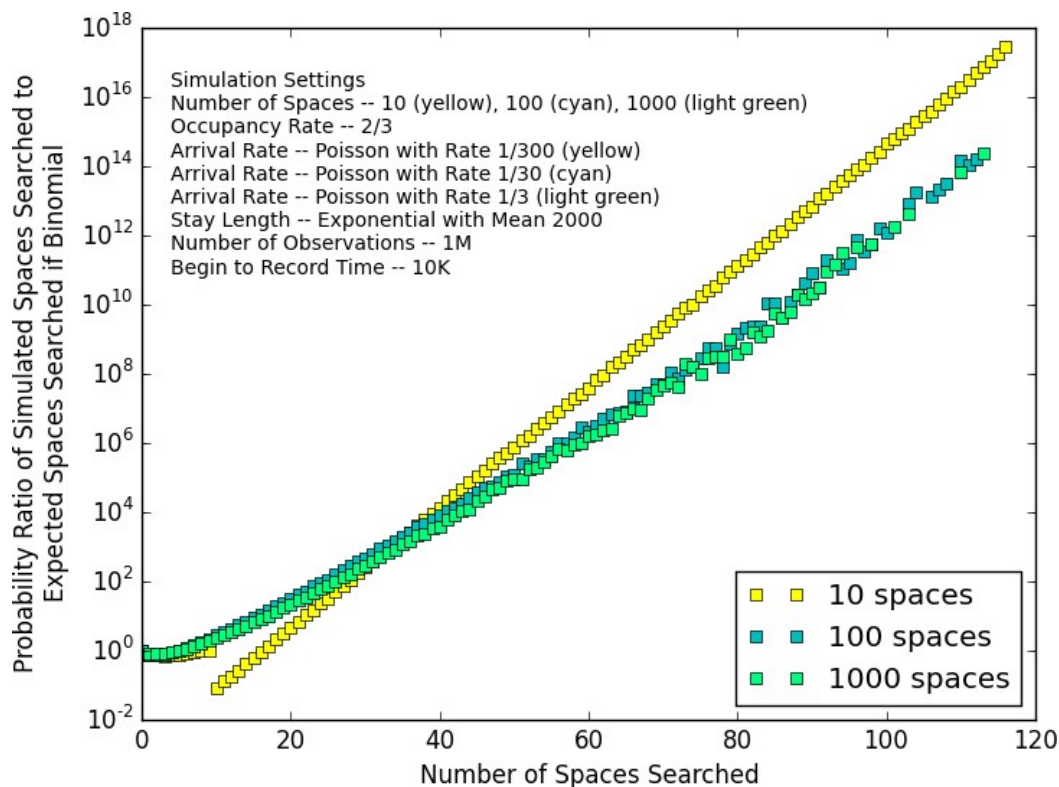


Figure 13: Ratio Plots: Number of Parking Spaces = 10, 100, and 1000 and Expected Occupancy Rate = 2/3

Let us consider the parking system when, after a long series of statistical accidents, it finds itself in a situation where all 10 parking spaces are occupied and 5 cars are cruising for parking. Each of the 10 parking spaces is vacated at the rate $1/2000$, for a Poisson death rate of occupied parking spaces of $1/200$. The Poisson entry rate is $1/300$. Thus, the parking logjam will eventually break up, but it may take a long time. From the perspective of a driver, the Poisson rate at which she will be the first to pass by a recently vacated parking space is $1/1000$. If the situation were to persist, the expected cruising-for-parking time for the driver would be 1000 (1/2 hour). But the situation will likely not persist. In the next increment of time, dt , the probability is $dt/200$ that the system will make a transition from 5 cars circling the block to 4, and $dt/300$ that it will make a transition from 5 cars circling the block to 6. We refer to parking as being saturated when a vacated parking space is taken almost immediately by one of the cars that is cruising the block. In this saturated state, since each of the cars cruising round the circle has the same probability of being the first to encounter the single parking space that has just been vacated, the queue discipline is random access, so that the parking system behaves like an $M/M/1/\infty/RSS$ queuing system. The common probability of encountering a parking space in an increment of time dt is $(P\mu/n)dt$, where $P\mu dt$ is the probability that some space is vacated so that $P\mu/n$ is the probability that a recently-vacated space is taken by a particular car among the n that are cruising for parking²⁸.

²⁸ The reader might be inclined to dismiss the "small-P" case as unrealistic. In most real-world situations, P is large, but also parking spaces are differentiated according to distance to the parker's destination, and many parkers are averse to walking long distances from their parking space to their destination. Consequently, many parkers do follow a circling-the-block strategy (often supplemented with a decision to garage park

Table 6 compares the moments of the cruising-for-parking time distribution for three situations. The first row displays the results for the central base case. The second row displays the results for a case that is identical to the base case except that the entry rate is doubled and the mean parking duration halved (while maintaining the negative exponential distribution of stay times), resulting in no change in the expected occupancy rate but a doubling of the turnover rate. The mean cruising-for-parking time is about 15% lower with the higher turnover rate, which is consistent with the corresponding ratio plot displayed in Figure 14, Panel A. Our tentative explanation is that an increase in the turnover rate reduces the persistence of the effects of extreme statistical accidents.

Parameters	Mean	Variance	Skewness	Fisher Kurtosis
Base Case	4.164	33.32	3.589	27.71
Base Case but Doubling the Turnover Rate	3.611	24.04	3.823	31.57
Base Case but Constant Stay Time	4.159	33.34	3.782	36.46

Table 6: Moments of the Distribution of Cruising-for-parking Times in Normalized Time Units

Notes: For each case, the numbers are from a single simulation with 10^6 cars.

after a certain number of unsuccessful circles of the block), even though they should recognize that the probability of finding a parking space on the second circuit of the block is typically much smaller than on the first circuit.

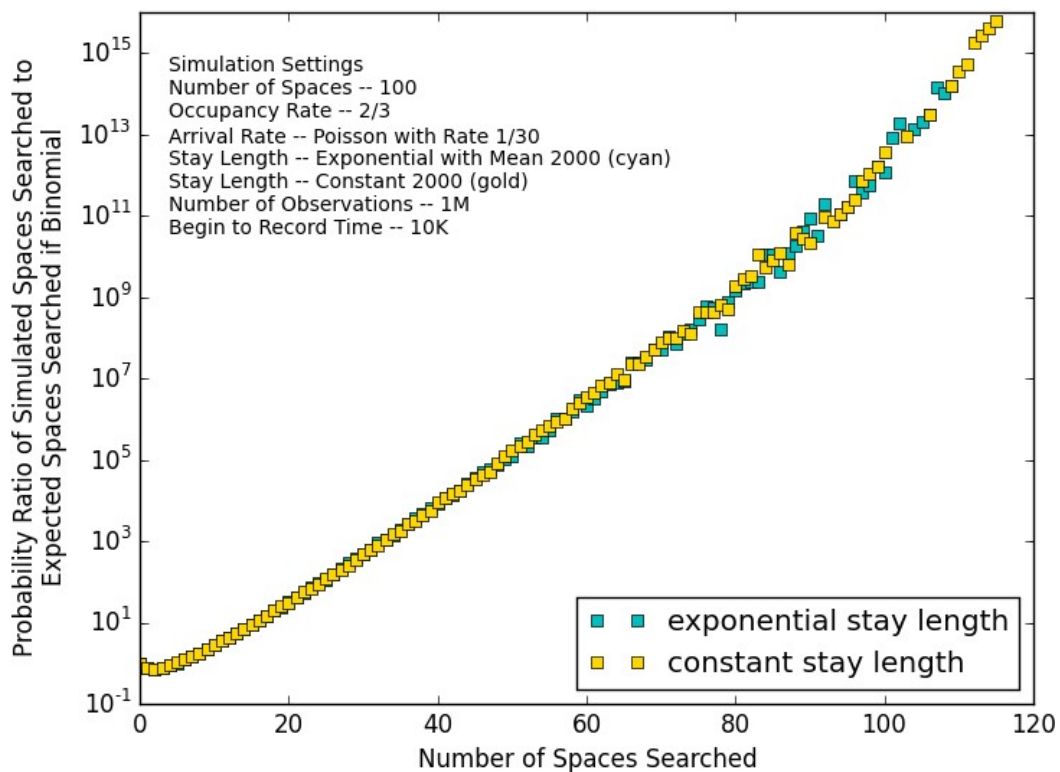
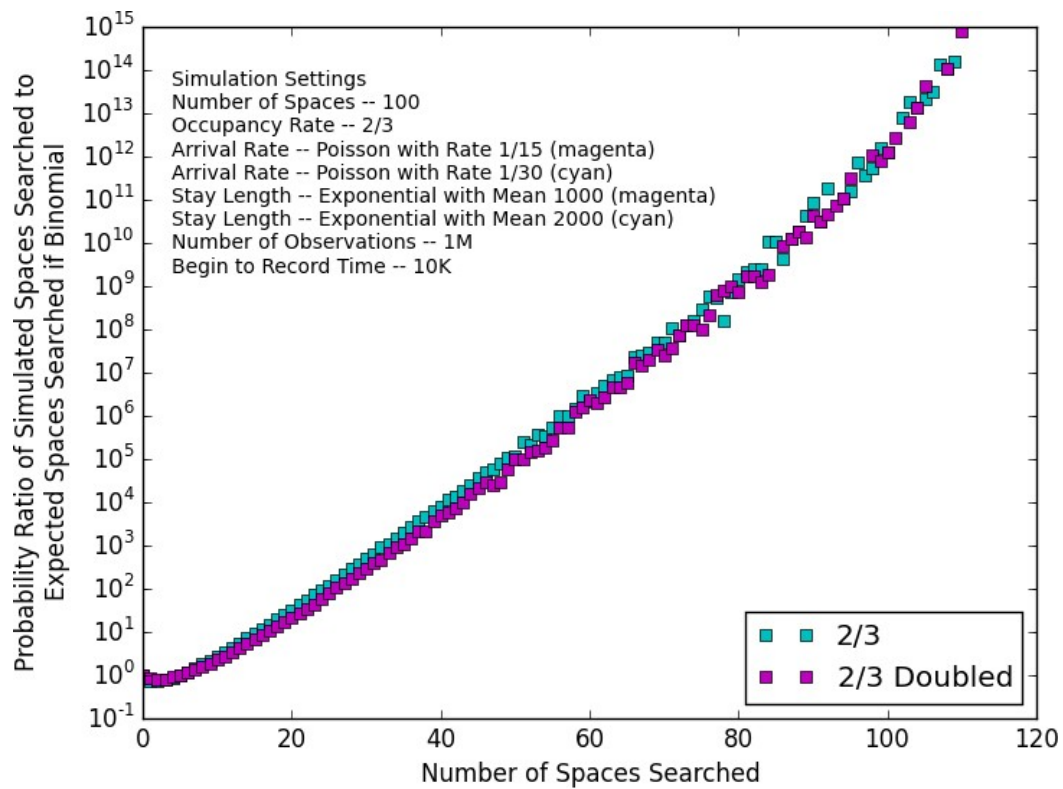


Figure 14: Ratio Plots

Notes: Panel A: Base case compared with base case but double the turnover rate.

Panel B: Base case compared with base case but constant parking duration.

The third row displays the results for the case that is identical to the base case except that the parking duration is constant at one hour rather than being negative exponentially distributed with a mean of one hour (in terms of the queuing theoretic notation introduced earlier, $B = D$ (deterministic) rather than $B = M$ (Markovian)). The mean of the cruising-for-parking time with a constant parking duration is not significantly different from that with a stay time with the same mean that is negatively exponentially distributed, which is visually evident in the corresponding ratio plot in Figure 14, Panel B. Our tentative explanation is as follows: On one hand, with a constant stay parking duration, parking is less likely to become saturated because the source of stochasticity deriving from variation in parking durations is absent; on the other hand, if parking does become saturated, it is more likely to persist.

At the risk of oversimplification, it appears that the parking system we have examined has four phases. In the first phase, the expected parking occupancy rate is modest and the system behaves in much the way predicted by the binomial approximation. The binomial approximation still severely underestimates the fatness of the right tail of the distribution of cruising-for-parking times but there is little probability weight there. In the second phase, the expected occupancy rate rises to the point that, via Jensen's Inequality, stochastic fluctuations in arrivals and exits generate stochastic fluctuations in the realized vacancy rate that raise the expected cruising-for-parking time substantially above that predicted according to the binomial approximation. Bunching is a related phenomenon,

reflecting localized variations in the realized vacancy rate. In this phase, the circling-the-block effect and the competition effect are unimportant. In the fourth phase, by statistical accident, parking becomes saturated, with an accumulation of cars circling the track hoping to be the first to pass by a parking space that has just been vacated. In this fourth phase, the circling-the-block effect and the competition effect dominate the behavior of the parking system, which is relatively easy to describe. The third phase, which lies between the second and the fourth phases, is the most complicated. Even though parking is not saturated, there is still a sufficient accumulation of cars cruising for parking that the circling-the-block becomes significant. If stochastic realizations are favorable, the system moves out of this phase into phase two; if they are unfavorable, the system moves into phase four.

6. *Concluding Comments*

This paper reported on a voyage of discovery into cruising for parking. It examined cruising for parking in about the simplest context possible -- cruising for parking at a constant speed around a circle with evenly spaced parking spaces in a stochastic steady state, with a temporally and spatially invariant Poisson entry rate and negative exponentially distributed parking stay times. Even such a simple model appears to be analytically intractable. To investigate its properties, we employed stochastic simulation modeling without resort to any *ad hoc* assumptions. This is in contrast to most previous work on cruising for parking which has employed what we termed the "binomial approximation", that the probability that each parking space is vacant equals the mean vacancy rate, independent of history and the current state of parking around the track.

The exploration generated considerable data. Even though we analyzed them with only crude statistical tools, our results make a compelling case that, except in low occupancy rate situations where cruising for parking is not a problem, the binomial approximation causes significant, and in high occupancy situations, severe, underestimation of expected cruising-for-parking times. The exploration also uncovered many results that to us at least were unexpected, and raised more questions than it answered.

There are a number of obvious directions for future research. We proceeded on the principle that it is important to understand the basics before adding complications in the direction of realism. Along these lines, an obvious direction is to apply more sophisticated statistical tools to analyze the data generated by both the simulation runs we undertook (which are accessible via the hyperlink math.ucr.edu/~parker/CruisingForParking/) and similar runs. For queuing theorists, an obvious direction is to apply the full arsenal of queuing theory to our model, or to simplified variants of it, attempting to obtain analytical results regarding the properties of our model and better analytical approximations for the distribution of cruising-for-parking time.

The next bold step forward will be to generalize the paper's model to an isotropic, two-dimensional space. This generalization is challenging because it qualitatively changes the nature of the problem. In the one-dimensional problem we analyzed, a driver is simply a cellular automaton that keeps on driving round the circle until finding a vacant

parking space. In the two-dimensional problem, in contrast, the driver can adopt a wide variety of strategies in searching for a curbside parking space. Attempting to solve for a Bayesian Nash equilibrium is unrealistically ambitious. A simple approach is to compare the equilibria when all drivers adopt the same strategy. A more sophisticated approach is to apply evolutionary game theory, looking for an evolutionarily stable mix of strategies. Following Arnott (2014), taking into account that most drivers have a specific destination and a desired arrival time, and will decide when to depart and when to start cruising for parking trading off in-transit travel time, cruising-for-parking time, walking time, schedule delay, and parking costs, would improve the model's realism²⁹.

Because of the difficulty of treating non-stationary dynamics, the analysis of most queuing models focuses on the steady state. But it is important to analyze cruising for parking in the context of rush-hour congestion dynamics since the expected occupancy rate varies systematically over the rush hour (indeed Geroliminis, 2015, does this under the binomial approximation). Practically, the best that can be hoped for is simulation models that endow cellular automata with some degree of sophistication rather than full rationality.

The paper's model assumes that drivers just keep on cruising for curbside parking until they find a vacant space. But realistically cruisers for parking develop stopping rules for

²⁹ Arnott (2014) solves for the optimal target (mean) curbside occupancy rate, conditional on the form of the distribution of cruising-for-parking times (which depends on the mean occupancy rate) It would be interesting to use the distributions of cruising-for-parking obtained in this paper's simulations in that paper's model.

a combination of reasons: continuing cruising for parking becomes increasingly frustrating; continuing cruising for parking will result in being increasingly late for an engagement; and continuing cruising for parking results in an accumulation of information that a change of action is in order. What those stopping rules are depends on the alternatives -- balking (just returning home), changing the sequencing of the day's activities, parking in a garage³⁰, and changing the search location. If stopping rules are not considered, cruising-for-parking models will consistently overestimate the probabilities of extreme cruising-for-parking times.

It is important to take account of spatial inhomogeneity. Much of the spatial inhomogeneity most relevant to cruising for parking is highly localized, pointing to a network simulation approach using complete street networks.

Another obvious direction for future research is to provide an integrated treatment of cruising for parking and traffic congestion. Cruising for parking affects traffic congestion, and traffic congestion affects the speed at which cars cruise for parking. Some work along these lines has been undertaken in Gerolominis (2015).

Thus far the paper has made scant mention of economics. In cruising for parking, a driver not only increases the expected cruising-for-parking time of other cruisers for parking but also contributes to traffic congestion. Parking pricing should account for both these

³⁰ While the paper has cast cruising for parking in the context of curbside parking, cruising for parking in parking lots and in parking garages is also important.

externalities. Capacity is the other main category of parking policy. How should the rules proposed (e.g., in Arnott, Inci, and Rowse, 2016) to choose the amounts of curbside, public garage, and private garage parking be modified to account for the more sophisticated treatment of cruising for parking that this paper investigates?

The simulation results reported in the paper indicate how much extreme stochastic events can affect expected cruising-for-parking times. An analogous phenomenon in the context of traffic congestion has recently been receiving considerable attention, with policy discussion focusing on how hypercongestion (situations in which traffic become so congested that traffic flow falls as traffic density rises) should be nipped in the bud through enlightened traffic management policies, such as diverting traffic via signal phase timing around locations where severe hypercongestion is imminent. How can the analogous phenomenon of near-gridlock in curbside parking in a downtown locale (in which a stock of cruisers for parking has accumulated) be avoided? The problem is made more difficult by the inability to distinguish between cars in transit and cars cruising for parking. Over sixty years ago Vickrey (1954) proposed (truly³¹) responsive curbside parking pricing, in which the price of curbside parking on a block is continuously adjusted so that at least one parking space is almost always available. At the time the policy was impractical, but now, with cell phone apps permitting curbside parking

³¹ Vickrey applied the term "responsive pricing" to pricing that adjusts automatically and in real time to stochastic realizations. Shoup uses the term differently, to refer to *ex ante* pricing that is adjusted periodically on the basis of average performance but not in response to stochastic realizations

reservations (Du and Gong, 2016), such a policy might be implementable without causing drivers excessive frustration.

In the context of cruising for parking, Levy *et al.* (2013) characterized the binomial approximation as "working with averages", in contrast to their simulation model, which took explicit account of stochastic fluctuations. They emphasized the importance of taking stochasticity into account, and this paper's simulations have underscored their message. The enlightened management of curbside parking needs to take into account temporally and spatially localized stochasticity. The same is true of downtown traffic congestion more generally. How to accommodate such stochasticity in the design of downtown transportation policy is an important general challenge for transportation researchers.

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Appendix 1: Source Code

Source code, available at <http://math.ucr.edu/~parker/CruisingForParking/>

```

from __future__ import division
import numpy as np

#model variables
#length of block
n = 100
#arrival rate
arrivalRate = 30
#stay duration
S = 2000
#stabilization cutoff. Dont record data before this timestep
sCutoff = 10000
#desired number of observations
numObs = 1000000
prob = 2/3
parameter = "2/3"

#Load arrays for reproducibility
initializationSettings =
np.load("RandomValuesForReproducibleRun/InitializationSettings"+str(parameter).replace('/',")+".numpy"
).tolist()
arrivalTimes =
np.load("RandomValuesForReproducibleRun/ArrivalTimes"+str(parameter).replace('/',")+".numpy").tolist()
stayLengths =
np.load("RandomValuesForReproducibleRun/StayLengths"+str(parameter).replace('/',")+".numpy").tolist()
arrivalLocations =
np.load("RandomValuesForReproducibleRun/ArrivalLocations"+str(parameter).replace('/',")+".numpy").toli
st()

#car object
class car:

    def __init__(self,
                 startingLocation,
                 startingTime,
                 currentTime,
                 carId,
                 startedSearching,
                 foundParking
                 ):
        self.startingLocation =startingLocation
        self.presentLocation =startingLocation
        self.startingTime = startingTime
        self.currentTime = currentTime
        self.carId = carId
        self.startedSearching = False
        self.foundParking = False
    def startinglocation(self):
        return self.startingLocation
    def presentLocation(self):
        return self.presentLocation
    def startedSeartching(self):
        return self.startedSearching

```

```

def foundParking(self):
    return self.foundParking
def currentTime(self):
    return self.currentTime
def repr_(self):
    return "Car id " \
        + str(self.carId) \
        + " started at " \
        + str(int(round(self.startingLocation))) \
        + " at time " \
        + str(self.startingTime) \
        + " Currently at " \
        + str(int(round(self.presentLocation))) \
        + " with found parking status " \
        + str(self.foundParking) \
        + " and started searching status " \
        + str(self.startedSearching)

#parking space object
class parkingSpace:

    def __init__(self,
                 location,
                 occupied,
                 emptyBy
                 ):
        self.location = location
        self.occupied = False
        self.emptyBy = 0

#objects required for simulation that are not model parameters
#array of searchtimes.
id = 0
searchTimes = []
occupancyRate = []
cars = []
currentlySearching = []
parkingSpaces = []
nextArrival = arrivalTimes.pop(0)
streetSnapshots = []

#array of cars start it with 15
for i in range(0,15):
    cars.append(car(arrivalLocations.pop(0),nextArrival,nextArrival,id,False,False))
    id +=1
    nextArrival = arrivalTimes.pop(0)

#array of parking spaces
for l in range(0,n):
    if(prob > initializationSettings.pop(0)):
        parkingSpaces.append(parkingSpace(l,True,stayLengths.pop(0)))
        print('starting with space '
              + str(l)
              + ' filled, it will be empty by '
              + str(parkingSpaces[l].emptyBy))
    else:
        parkingSpaces.append(parkingSpace(l, False, 0))
step = 0
while( len(searchTimes) < numObs ):

```

```

print("Currently on step "
      + str(step)
      + " with "
      + str(len(searchTimes))
      + " observations")
#if we run low on cars add more, also make sure they dont arrive too quickly.
if(len(cars) < 10):
    while(nextArrival - (step+1) < 1):
        cars.append(car(arrivalLocations.pop(0),nextArrival,nextArrival,id,False,False))
        id+=1
        nextArrival = arrivalTimes.pop(0)

#move cars that begin to search from cars array to currentlySearching
for c in cars:
    if((c.startingTime < step) and c.startedSearching == False):
        c.startedSearching = True
        currentlySearching.append(c)
        print("Car "
              + str(c.carId)
              + " has begun to search ")

#see if the cars in current cars are on empty spaces
for cs in currentlySearching:
    if(cs.foundParking == False) and (parkingSpaces[int(np.floor(cs.presentLocation))].emptyBy <
cs.currentTime):
        #found parking
        cs.foundParking = True
        #fill the space
        parkingSpaces[int(np.floor(cs.presentLocation))].occupied = True
        print("parking space "
              + str(np.floor(cs.presentLocation))
              + " is now taken ")
        #get a new exit time
        parkingSpaces[int(np.floor(cs.presentLocation))].emptyBy = (stayLengths.pop(0) +
cs.currentTime)
        #if we are far enough along record it
        if(step > sCutoff):
            searchTimes.append(step - cs.startingTime)
        #output for console
        print("Car "
              + str(cs.carId)
              + " found parking at time "
              + str(step)
              + " in location "
              + str(np.floor(cs.presentLocation))
              + " having searched for time "
              + str(step - cs.startingTime))
    else:
        #space was full, advance the car
        print("Car "
              + str(cs.carId)
              + " advanced from location "
              + str(cs.presentLocation)
              + " to location "
              + str((cs.presentLocation + 1)%n))
        cs.presentLocation = (cs.presentLocation + 1)%n
        cs.currentTime +=1

```

```

#cleanup - remove cars that have found parking, allow new cars to merge onto street in order
currentlySearching[:] = [x for x in currentlySearching if (x.foundParking == False)]
cars[:] = [y for y in cars if (y.startedSearching == False)]
if(len(currentlySearching) != 0):
    currentlySearching = sorted(currentlySearching,key = car.presentLocation)

currentSpacesTaken = 0
#empty out spaces that were not taken
for p in parkingSpaces:
    if(p.emptyBy < step) and (p.occupied is True) :
        p.occupied = False
        print("Space "
              + str(p)
              + " is now available")
    if(p.occupied == True):
        currentSpacesTaken +=1

#reord ambient data
if(step > sCutoff):
    occupancyRate.append([currentSpacesTaken/100,len(currentlySearching)])

step +=1

searchTimesToBeSaved = np.array(searchTimes)
np.save(parameter.replace('/',")+ '/' +parameter.replace('/',")+ "_searchTimes",searchTimesToBeSaved)
occupancyRateToBeSaved = np.array(occupancyRate)
np.save(parameter.replace('/',")+ '/' +parameter.replace('/',")+ "_occupancyRate",occupancyRateToBeSaved)

```

Appendix 2: Generated Simulation Data

There are two types of simulation programs available here -- random and reproducible.

The random programs allow the user to rerun our simulations with different stochastic realizations. The generic title of the random programs is Cruising_For_Parking.py.

Under this title is a set of files, each corresponding to a different case considered in this paper. The top of each file indicates the model parameters employed, with the comment #model variables above them. The random programs allow the researcher to generate multiple simulation runs for the same case, and hence to explore the impact of

microscopic stochasticity on stochastic aspects of the aggregate behavior of the cruising for parking system.

A reproducible program for a particular case differs from the corresponding random program only in that the stochastic realizations that were generated in the single reproducible simulation run reported on in the paper are used as data. The set of files for the reproducible program have the generic title

Cruising_for_Parking_ReproducibleXX.py. The XX indicates the case treated in each particular file. The reproducible programs will allow the research to check the results of the simulation runs reported in the paper, and to apply other statistical tests to those runs. For each case, there are two other files. One reports the occupancy rate in the first column and the number of cars searching in the second column for each time unit. The other reports the search time for each car. Their aim is to permit faster analysis of these central series. There is an additional file Random_Values_For_Simulation.py that was used to generate all random elements for the reproducible runs.

All the runs reported in this paper are reproducible, with two exceptions. Figure 8 was done with 1000 randomized simulations, running the simulation inside Cruising_For_Parking.py 1000 times and storing only summary statistics. Figures 9 and 10 are also pulled from a random run, and the street snapshots are saved as snapshots23.npy and snapshots56.npy.

The directory structure on the site is organized as follows

File	Purpose
Cruising_For_Parking.py	Random simulation
Cruising_For_Parking_ReproducibleXX.py	Simulation that will reproduce our work, where XX comes from the shorthand in the next table.
Moments_All.py	Summary statistics for all reproducible simulations
Random_Values_For_Simulation.py	This generated all random elements for the reproducible runs.
SnapshotsXX.py	Street snapshots taken from the random runs used to generate figures 9 and 10

The shorthand naming convention is as follows

Short Hand	Description
23	Occupancy Rate $2/3$. Stay lengths are exponentially distributed with mean 2000, entry rate is Poisson with mean $1/30$, track has 100 spaces.
23Const	Occupancy Rate $2/3$, Constant stay of 2000 time units
23D	Occupancy Rate $2/3$, Doubled entry rate halved stay length
23_10spaces	Occupancy Rate $2/3$, 10 spaces
23_1000spaces	Occupancy Rate $2/3$, 1000 spaces
56	Occupancy Rate $5/6$
110	Occupancy Rate $1/10$
1112	Occupancy Rate $11/12$

There are also corresponding folders for each line of the shorthand table. Each folder contains two files, which will have been generated by

`Cruising_For_Parking_ReproducibleXX.py`

File	Contents
<code>XX_occupancyRate</code>	A time series without fixed length, it recorded the occupancy rate in the first column and the number of cars searching in the second column for however long it took to observe 1 million search times.
<code>XX_searchTimes</code>	1 million observations of how long a car took to find parking.

BUNCHING IN CURBSIDE PARKING AROUND A CIRCLE

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This technical report documents work done on the second phase of the research project, "Towards Inferring Welfare Changes from Changes in Curbside Parking Occupancy Rates: A Theoretical Analysis Motivated by SFpark and LA Express Park", Department of Transportation Contract No. 65A0528. The authors would like to thank the Department of Transportation and Caltrans, which is co-sponsor on the project, for their financial support under this grant.

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BUNCHING IN CURBSIDE PARKING

Stimulated by the pioneering work of Donald Shoup (2005), there has recently been a sharp increase in the attention being paid to parking policy. For many years, Shoup has been advocating "cashing out (free or subsidized) parking" -- pricing parking to "clear the market". In most contexts, clearing the market entails setting the price so that the quantity supplied equals the quantity demanded, with no excess demand and no excess supply. In markets in which search is important, however, what constitutes efficient pricing is not obvious. Consider, for example, curbside parking, which is the focus of this paper. If the curbside parking fee is set high, there will be many vacant parking spaces, which entails an inefficient use of scarce curbside parking space. If the price is set low, there will be few vacant parking spaces. The utilization rate of curbside parking spaces will be high but drivers will expend their own time cruising for parking and then in walking from their parking spaces to their destinations, and increase other drivers' time by adding to traffic congestion while cruising for parking. Intuitively, the curbside parking fee should be set to optimally tradeoff these two types of costs.

Shoup's work has inspired a number of curbside parking experiments, the best known of which is SFpark (<http://sfpark.org>)¹. These experiments have two broad aims. The first is to operationalize the concept of "cashing out parking". The second is to ascertain the effectiveness of cashing out free or underpriced curbside parking in reducing the costs associated with cruising for parking. Shoup (2006) originally proposed that curbside

¹ Garage parking, which SFpark also treats, will be ignored in this paper.

meter rates be set *ex ante* by block and by time of day to achieve a common 85% target average curbside parking occupancy rate. Block by block and time period by time period, SFpark has been adjusting meter rates up and down so as to achieve what they perceive to be the optimal tradeoff between unutilized capacity and cruising-for-parking related costs. They have tended to choose target average curbside occupancy rates that are substantially lower than the 85% originally proposed by Shoup.

Arnott (2014) presented a conceptual framework for determining the optimal target (average) curbside parking occupancy rate in a (stochastic) steady state in an isotropic space with risk-neutral drivers, only curbside parking, and no traffic congestion. The optimal (average) curbside parking occupancy rate is determined by three relationships. The first relates expected marginal social trip cost to the average curbside parking occupancy rate (referred to hereafter simply as the occupancy rate); the second relates expected throughput to the occupancy rate; and the third relates drivers' marginal willingness to pay for a trip to throughput (the demand function). Combining the first two relationships gives a relationship between expected marginal social trip cost and throughput. The optimal level of throughput is that for which the expected marginal social cost of a trip equals the marginal social benefit, which is taken to be the marginal willingness to pay for a trip. The optimal curbside meter rate is that which achieves the optimal level of throughput.

Relating expected marginal social trip cost to the occupancy rate requires, *inter alia*, relating expected cruising-for-parking time to the occupancy rate. *The modest goal of*

this paper is to explore the relationship between expected cruising-for-parking time and the occupancy rate for the stochastic steady state of cars cruising for parking round a circle. While the goal is modest, some of the qualitative insights gained should have broader application, as we shall discuss later.

To our knowledge, all previous theoretical work relating expected cruising-for-parking time to the occupancy rate have modeled cruising for parking as entailing random draws with replacement from a binomial distribution, where a draw entails the search of a parking space. We term this the *binomial approximation*.² Where q is the occupancy rate, the probability of finding the first vacant space on the n th draw³ is $q^{n-1}(1 - q)$, and the expected number of draws⁴ is $1/(1 - q)$. Thus, for example, with an occupancy rate of 80%, the expected number of curbside parking spaces searched⁵ is five. Arnott and Rowse (1999) make this binomial assumption, recognizing it as an approximation.

² Since the probability distribution of the number of parking spaces searched is the geometric distribution, this might properly be called the "geometric approximation", but the term binomial approximation is more informative since "geometric" has a wide variety of meanings.

³ The probability of finding the first vacant space on the first draw (i.e., first parking space) is $1 - q$. The probability of finding the first vacant space on the second draw is the probability of finding the first parking space occupied, q , times the probability of finding the second space vacant, conditional on finding the first parking occupied, which, under the assumption of random draws with replacement, equals the unconditional probability of finding the second space vacant, $1 - q$. Etc.

⁴ Let S be the expected number of draws. $S = (1)(1 - q) + (2)[q(1 - q)] + (3)[q^2(1 - q)] + \dots = (1 - q)\{1 + 2q + 3q^2 + \dots\}$. Multiplying both sides by q yields $qS = (1 - q)\{q + 2q^2 + 3q^3 + \dots\}$. Subtracting qS from S yields $(1 - q)S = (1 - q)\{1 + q + q^2 + \dots\}$. Since the value of the infinite sum in the curly brackets is $1/(1 - q)$, $S = 1/(1 - q)$.

⁵ When we say "the expected number of parking spaces searched", we include the last search, in which the first vacant space is found.

Subsequently, other researchers, including Anderson and de Palma (2004) and Geroliminis (2015) have made the same assumption.

Levy, Martens, and Benenson (2012) consider a situation in which drivers search for parking in a residential neighborhood on their return from work, and in which therefore the occupancy rate increases as the evening proceeds. They compare the expected number of parking spaces searched under the binomial assumption with the realized average number of parking spaces searched in their simulation model, PARKAGENT, as functions of the realized occupancy rate. They find that, when the realized occupancy rate is high, the simulated average number of parking spaces searched is considerably higher than the expected number under the binomial approximation. Though their analysis is not steady state, and though their conclusion rests on the soundness of their simulation model, their paper succeeds in raising doubt about the accuracy of the binomial approximation applied to the stochastic steady state, especially at high occupancy rates.

There are further reasons to doubt the accuracy of the binomial approximation.

1. The binomial approximation takes the occupancy rate as being constant over time. But even in a stochastic steady state, in which entry occurs according to a time-invariant Poisson process, the local occupancy rate would fluctuate because of the stochasticity of demand at the local level.
2. In the context of SFpark, the occupancy rate is measured as the arithmetic average over a time period during which the expected occupancy rate can be expected to change.

Since the expected number of parking spaces searched is a convex function of the contemporaneous occupancy rate, the expected number of parking spaces searched over the time period exceeds that which would occur if the occupancy rate had been at its average over the entire period. To illustrate, suppose the occupancy rate is 65% half the time, and 95% the other half, so that the expected occupancy rate is 80%. Under the binomial approximation, the expected number of parking spaces searched is $0.5(1/0.35) + 0.5(1/0.05) = 1.4286 + 10 = 11.4236 > 5 = 1/0.20$. An analogous argument applies when the occupancy rate is measured over an area that is spatially inhomogeneous, for example including both arterials and back streets, or including areas where congestion is more or less heavy.

3. The expected cruising-for-parking times generated by the binomial approximation square neither with experience nor with policy discussion. In particular, the expected cruising-for-parking times it generates seem consistently too low. Experience suggests that on blocks where the *average* occupancy rate is, say, 80%, it may nonetheless be difficult to find a curbside parking space on that block with reasonable frequency; in contrast, according to the binomial approximation, the expected number of spaces searched before finding a vacant space is only five. Policy discussion in Shoup (2005) and elsewhere indicates that cruising-for-parking is perceived to be a serious problem in downtown areas, and yet the expected cruising-for-parking times generated under the binomial assumption are modest, except as the occupancy rate approaches one⁶.

⁶ Table 11-5 in Shoup (2005) reports that average cruising for parking time over the 16 studies of cruising for parking he located was 8.1 minutes, about 500 seconds. Our calibration for the numerical simulations, presented below, imply that it takes 1.8 seconds to travel from one parking space to the next. Applying this figure implies that the

4. The binomial approximation is based on the assumption that the occupancy probabilities of adjacent parking spaces are statistically independent. But common sense and experience suggest that the occupancy probabilities of adjacent parking spaces are both spatially and temporally correlated. Suppose, for example, that a restaurant that is normally not crowded is holding a banquet. This will affect the occupancy rate not only of the curbside parking space immediately adjacent to the restaurant but along the entire block and likely further along, as guests who find the parking space adjacent to the restaurant occupied continue to search for parking. According to this line of reasoning, occupied parking spaces are likely to be *bunched*.

5. Intuitively, the optimal curbside occupancy rate might be different in the morning rush hour, when the occupancy rate is increasing (a driver might be tempted to grab a parking space while it is still available even if it is at an inconvenient location), than in the evening rush hour when the occupancy rate is decreasing (a driver might pass by an inconvenient parking space, confident that he will be able to find a parking space nearer to his destination). Intuitively too, the optimal occupancy rate may depend on the turnover rate, or some other aspect of the probability distribution of parking durations.

In this paper, we consider a stylized model that abstracts from most of these complicating considerations. Parking spaces are uniformly distributed around a circle. Drivers arrive according to a time-independent and space-independent Poisson process. They cruise around the circle in one direction at an exogenous speed, and take the first vacant parking

expected number of parking spaces searched is 278. Under the binomial approximation, this implies an occupancy rate of 99.64%.

space they encounter⁷. The issue is whether, in stochastic steady state, the time- and space-independent Poisson arrival process generates a time- and space-independent vacancy generation process. If it does, the binomial approximation is sound. Put alternatively, even though the arrival process is time- and space-independent, does bunching occur in space (spatial autocorrelation), in time (temporal autocorrelation), or in both?

We originally explored exact analytical solution, but had no substantive success. We then had the choice between investigating analytical solution under approximating assumptions or proceeding to simulation. We decided to start with simulation since, without an "exact" solution, we would have no way of judging how accurate were the analytical solutions under approximating assumptions.

⁷ In earlier unpublished research, Arnott and Qiong Tian, who was visiting the University of California, Riverside from the Beijing University of Aeronautics and Astronautics, attempted to solve a version of the model with discrete space (evenly spaced parking spaces around a circle) and time from the perspective of queuing theory. Each parking space is viewed as a server. The state of the system is described by the states of each of the sequential servers. A server may be vacant, occupied with no queue, or occupied with a queue of varying and potentially infinite length. To simplify, Arnott and Qiong assumed that parking durations are negative exponentially distributed, so that the probability that a car exits a parking space is independent of how long it has been parked there. Between time periods, some cars enter the system according to a space- and time-invariant Poisson process, a fraction of parked cars exit the system with the rest staying in their parking spaces, and the remaining cars, which are in queues at servers, advance one parking space, which corresponds to cruising for parking. The aim was to derive the distribution of cruising-for-parking times from the stationary distribution of states of the system. Having been unsuccessful in obtaining analytical results, even with approximations, Arnott and Qiong started numerical simulation. When Qiong returned to China, Arnott and Derek Qu, then a graduate student in computer science at the University of California, Riverside and now at Google, continued the work, with some success.

Our basic finding is that the binomial approximation is a bad one. With the modest expected occupancy rate of $2/3$, the simulated expected cruising-for-parking time is 3.08 time units, while that obtained under the binomial approximation is 2.5 time units, a ratio of 1.252. As the occupancy rate increases, so too does the ratio of the simulated expected cruising-for-parking time to that obtained under the binomial approximation. With an expected occupancy rate of $11/12$, the simulated expected cruising-for-parking time is 58.73 time units, while that under the binomial approximation is 11.5 time units, a ratio of 5.106

In section 2, we present the results from the base simulation run, discussing methodological issues relating to describing the outcome statistically. In section 3, we present the results for other simulation runs. In section 4, we provide some discussion, and in section 5 we conclude. A technical appendix contains a verbal and then a verbal flowchart description of the simulation algorithm employed, and a link to the simulation program.

2. The Base Case Simulation

100 parking spaces are evenly spaced on the circumference of a circle. Each space is treated as a point. Speed is measured in units such that a car takes one time unit to travel the distance between adjacent parking spaces. For the sake of concreteness, we may imagine that the distance between adjacent parking spaces is 21.12 ft, so that the circumference of the circle is 0.4 mile, and that cruising-for-parking speed is 8 mph.

Then a car travels round the circle in $0.05 \text{ hr} = 3 \text{ minutes} = 180 \text{ seconds}$. Since there are 100 parking spaces around the circle, it takes a car 1.8 seconds to travel from one parking space to the next, so that a time unit equals 1.8 seconds.

Cars arrive on the circle according to a time- and location-independent Poisson process at a rate of $1/30$ per time unit. The time-independent stay length or parking duration of a car is negative exponentially distributed with mean 2000 time units, the same as all locations. The expected occupancy rate in stochastic steady state can be calculated as the expected total time parked in all parking spaces per unit time divided by the maximum total time parked in all parking spaces per unit time (100). Since the entry rate and time parked are statistically independent, expected total time parked in all parking spaces per unit time equals the expected entry rate to parking spaces ($1/3$) multiplied by the expected time parked per parking space (2000). Thus, the expected occupancy rate is $2/3$.

To allow for the possibility (which turned out to be the case) that there is substantial temporal autocorrelation in the realized occupancy rate, each simulation run has 10^6 cars pass through the system. Since the entry rate is $1/30$ car per time unit or $(1/30) \div 1.8 = 1/54$ car per second or $(2/3) \times 10^2$ cars per hour, each run simulates $(3/2) \times 10^4 = 15,000$ hrs of traffic. The starting point of each simulation had no cars in the system. To avoid having the initial conditions affect the calculated distribution of cruising-for-parking times, we started to record results after 10,000 time units = 18,000 seconds = 5 hrs. In other words, we allowed the system 5 hours to settle into the stochastic steady state before starting to record results.

Throughout the paper, we record results in terms of the number of curbside parking spaces searched. It will be helpful to relate this to cruising-for-parking time. When a car first enters, the expected distance traveled before encountering a parking space is 0.5 distance units, which, according to our normalization, takes 0.5 time units. With a single successful search, the expected cruising-for-parking time is 0.5 time units. With two searches, the expected cruising-for-parking time is 1.5 time units, 0.5 expected time units to reach the first parking space, and then 1.0 time units to travel from the first to the second parking space. Thus, in time units, the expected cruising for parking time equals the number of parking spaces searched minus 0.5.

Under the binomial approximation, with an occupancy rate of $2/3$ the expected number of curbside parking spaces searched before finding a vacant parking space is 3. Thus, the expected cruising-for-parking time is 2.5 time units or 4.5 seconds.

How should the simulated distribution of cruising-for-parking times be compared to that calculated under the binomial approximation? Since we do not know the functional form of the proper distribution, which the simulated distribution provides an estimate of, non-parametric statistics are appropriate. We shall compare the distributions using three types of non-parametric methods: moments of the distributions, comparison of the probability density functions and cumulative distribution functions, and ratio plots.

Table 1 compares the first four moments of the two distributions⁸. The simulated distribution is for a single run. The results are consistent with the simulated distribution having a fatter right-hand tail than the binomial distribution.

	Mean	Variance	Skewness	Kurtosis
Simulated	3.579	25.99	4.596	42.61
binomial approximation	3.0	6.0	2.041	6.167

Table 1: Moments of the probability distribution of the number of parking spaces searched up to and including the first success, simulated and binomial approximation

Notes:

1. Numbers are presented to the fourth significant digit.
2. Under the binomial approximation, the probability distribution of the number of parking spaces searched. The formulae for the mean, variance, skewness, and kurtosis of the geometric distribution are $1/(1 - q)$, $q/(1 - q)^2$, $(1 + q)/q^{1/2}$, $6 + (1 - q)^2/q$, where q is the occupancy rate (wikipedia. Geometric distribution).

Figure 1 gives the probability density function and the cumulative distribution function for the simulated and approximated distributions. The negative exponential function with parameter $1 - q$ is the continuous analog of the geometric distribution with success probability $1 - q$. Only to facilitate visual comparison of the two distributions, the pdf and the cdf for the negative exponential distribution with parameter $1/3$ are given instead of the corresponding pdf and cdf for the geometric distribution. Panel A gives the pdfs and Panel B the cdfs. Again, the figures are consistent with the simulated distribution having a fatter tail than the binomial/negative exponential distributions.

⁸ It should be noted that, whatever the distribution of the number of parking spaces searched, up to and including the first vacant space, the probability that the first parking space searched is vacant is simply the vacancy rate or one minus the occupancy rate.

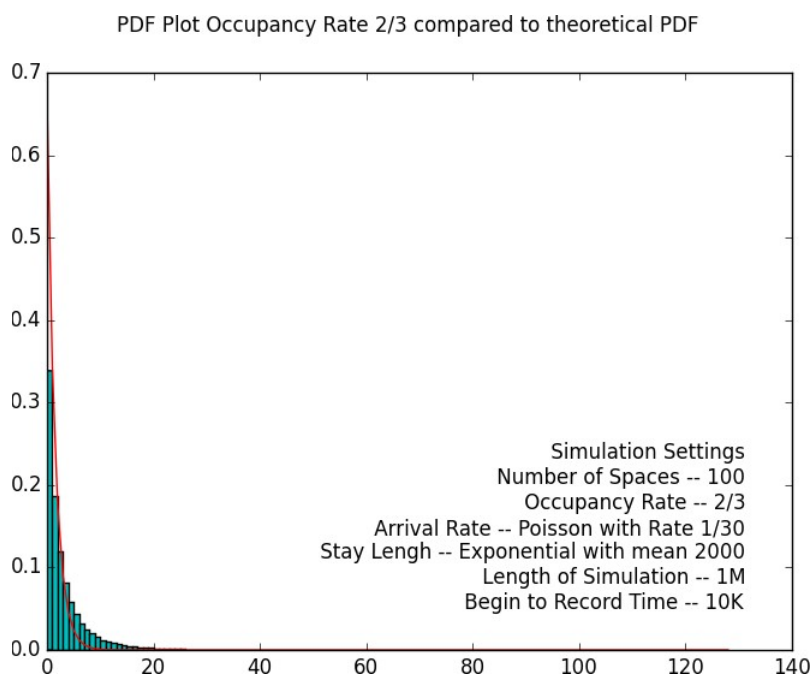
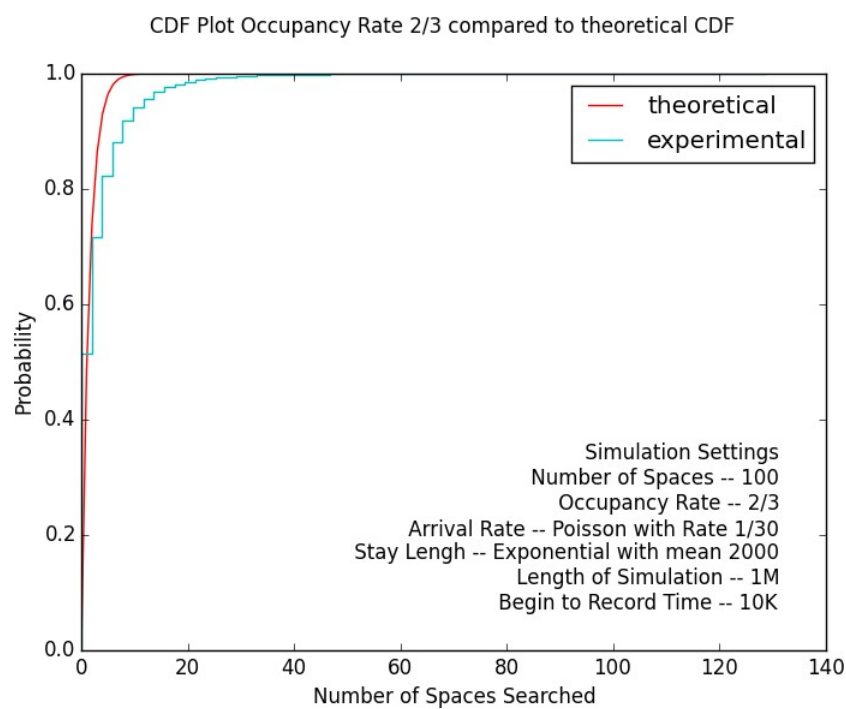


Figure 1: The cdf and pdf for the approximated distributions

1. Only to facilitate visual comparison of the two distributions, the pdf and the cdf for the negative exponential distribution with parameter $1/3$ are given instead of the corresponding pdf and cdf for the geometric distribution.

Ratio plots provide an insightful way of examining the fatter tail in the simulated distribution than that in the geometric distribution. Each data point gives the ratio of the simulated probability that the number of parking spaces searched equals x to the corresponding geometric probability. Note that the y-axis is on a power ten scale. Figure 2 provides the ratio plot for the example under consideration. From the Figure, it is easy to see that the simulated distribution has much fatter tails than the corresponding binomial distribution. The ratio of the probabilities initially decreases⁹, then increases from a ratio of about one for 6 parking spaces searched, to about 100 for 27 parking spaces searched, to about 10,000 for 45 parking spaces searched, and then to about 1,000,000 for about 52 parking spaces searched. These numbers can be so large since the corresponding probabilities with the geometric distribution are so small. For example, in the base case, with the geometric distribution the probability that 45 parking spaces are searched is $(2/3)^{44}(1/3) = 0.5955 \times 10^{-10}$, so that the correspond simulated probability is

⁹ Since the sum of the probabilities equals one for both distributions, the fatter tail for the simulated distribution requires that the probability ratio of the number of parking spaces searched be less than one for some numbers of spaces searched. Furthermore, the probability ratio must equal 1.0 for a single parking space searched.

of the order of 10^{-6} . Note that, overall the simulation period, the maximum number of parking spaces searched was 58.

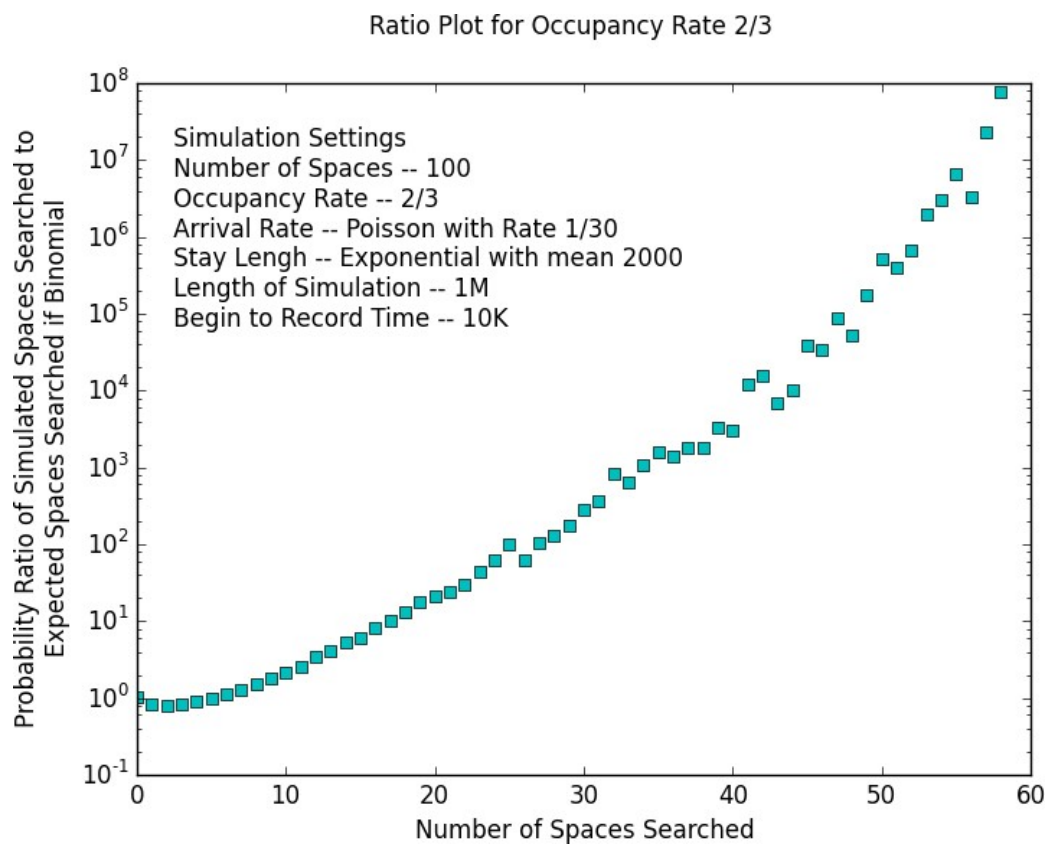


Figure 2: The ratio of the simulated probability that the number of parking spaces searched equals x to the corresponding probability for the geometric distribution.

Figure 2 especially provides a compelling visual argument that the binomial approximation is a bad one, and should be avoided, even for the modest occupancy rate of $2/3$. This raises two obvious questions, the answers to which are obviously related. First, what is the exact distribution of cruising-for-parking times around the circle that the simulated distribution approximates? Second, why are the tails of the simulated distribution so much fatter than those of the geometric distribution?

As noted earlier, we have been unsuccessful in providing an answer to the first question, even though the "data-generating process" is stationary and known exactly. We suspect that analytical solution is either intractable, or if not intractable, at least beyond our mathematical capabilities to obtain. We suggest that a promising way to proceed is to run a large number of simulations similar to the ones that we have done, several for each set of parameter values, for a wide range of sets of parameters values, and then to fit flexible functional forms to estimate statistics of interest, such as moments of the distribution as functions of such parameters as the expected occupancy rate and the number of parking spaces. We have not done this for two reasons. First, we feel that the paper has succeeded in making the *qualitative* point that the distribution of the number of parking spaces searched prior to finding a vacant space is not well approximated by the binomial function (in other words, does not conform well to the binomial approximation). Second, since the model of this paper considers only a one-dimensional isotropic space, and would therefore not be useful in practical application, detailed quantitative analysis of the distribution is not justified. In the next stage of our research, which entails search for

parking on a Manhattan grid, the model may be sufficiently realistic to warrant detailed quantitative analysis of its solution properties. For the same reasons, we judged it not to be a good use of time to investigate whether the right tail of the simulated distribution of the number of parking spaces searched is well described by extreme value distributions.

We have given considerable thought to answering the second question. We did some preliminary work investigating the serial correlation or autocorrelation of the occupancy rate at individual parking spaces, with time steps of the order of the mean of a parking duration to examine temporal persistence. We also considered applying standard spatial statistics, such as Moran's I-statistic, to examine the spatial correlation between adjacent parking spaces, as well as some form of kernel estimation to spatially smooth the data at different levels of spatial resolution in order to investigate the spatial clustering of occupied parking spaces¹⁰. Pursuing spatial clustering further, we also thought about applying the literature in mathematical physics on self-organizing systems. Upon further reflection, however, we realized that the bunching of occupied parking spaces occurs not just in time or in space but in space-time, and we were unable to find a literature on the subject. Furthermore, while the bunching of occupied parking spaces exhibits some of the characteristics of self-organizing systems, it is different in that the bunches or clusters in

¹⁰ Arnott recalls, as a high school student, looking at an onion through a microscope. At some magnifications structure was evident, while at others the image was a blur. Tomoya Mori has for many years been investigating the clustering of firms by industry in Japan (see Mori and Smith, 2015, for discussion of the most recent product of their long-term research program). He observes a similar phenomenon. Similarly here, one expects to observe structure at some spatial resolutions and disorganization at others.

Another possible approach is Fourier analysis. According to this approach, sine waves of different frequencies and amplitudes are generated by the shock anomalies, and interact, sometimes offsetting and sometimes reinforcing one another.

space-time do form but then they dissolve and reform elsewhere in complex ways. We plan to investigate literature in applied mathematics/mathematical physics that considers such problems, semi-chaotic systems, which exhibit some elements of chaos and some elements of self-organization.

Our inability to get a conceptual handle on the bunching of occupied parking spaces frustrated us for two different reasons. First, our model's data-generating process is so simple that intuition suggests that its spatial-dynamic behavior should be predictable if looked at through the appropriate lens. It may be, but if it is, then we were unsuccessful in finding the appropriate lens. Second, everyday experience provides intuition for the bunching of occupied spaces. Suppose that, completely by chance, even though the average occupancy rate is modest, all the parking on a particular block becomes occupied. Drivers who otherwise would have parked on that block continue searching, propagating that anomaly in space-time. Furthermore, other drivers might choose to circle that block, slowing down the rate at which the anomaly dissipates. Even though the phenomenon is part of everyday experience, it seems difficult to characterize.

The furthest we succeeded in going was to provide a picture of bunching in space-time, which is shown in Figure 3 (it looks like the bar code used on boarding passes). The x-axis is the spatial axis, with each number indexing a particular parking space. Moving vertically provides snapshots of the occupancy/vacancy pattern over time of a particular parking space every so many time units. The y-axis is the time axis, with each cell

representing a 1000th time step or units, or one half hour. If one row on the y-axis were to describe the parking situation at 9:00 am, then the row above would describe the parking

Snapshots of the street every 1000 time steps with occupancy rate 2/3
White is occupied Black is vacant

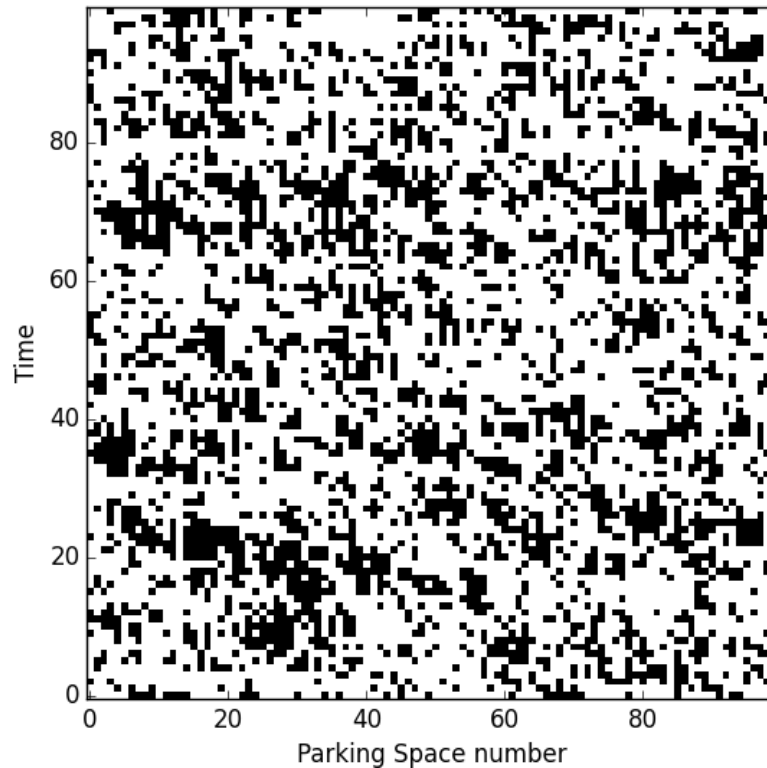


Figure 3: Bunching: Evolution of occupied and vacant parking spaces in space-time
(base case, occupancy rate 2/3)

Notes:

1. The x-axis is the spatial axis, with each number indexing a particular parking space. Moving vertically describes the occupancy/vacancy pattern every so many time units of a particular parking space. The y-axis is the time axis, with each cell representing a 1000th time step or unit, or one half hour. Moving horizontally describes the occupancy/vacancy pattern over space at a particular point in time.
2. The parameters, including the begin-to-record time, are the same as in the base case.
3. Since each time step corresponds to one-half hour, the Figure describes the evolution of parking around the circle over a 50 hr period.

situation at 9:30 am. Moving horizontally describes the occupancy/vacancy pattern over parking spaces at a particular point in time. The bunching of both occupied spaces (white) and vacant spaces (black) in space-time is very obvious to the naked eye, but even to describe satisfactorily it would require the development of new statistics.

The reader may have doubted our wisdom in starting our analysis of cruising for parking by examining cruising for parking round a circle, since almost all cruising for parking occurs in two-dimensional space. Imagine, however, trying to describe, let alone explain, a diagram similar to that of Figure 3, but for a two-dimensional Manhattan network, which would result in a three-dimensional version of Figure 3. Even for a one-dimensional isotropic space (the outside of a circle) and for the stochastic steady state, cruising for parking is evidently a complex phenomenon.

3. *Further Simulations*

In this section, we present simulation results obtained by altering parameter values one at a time.

3.1 Altering the arrival rate, and hence the occupancy rate

Table 2 gives the moments of the number of curbside parking spaces searched before finding a vacant parking space for three Poisson arrival rates: $1/30$ in the base case, which generates an occupancy rate of $2/3$; $1/24$, which generates an occupancy rate of $5/6$; and

11/240, which generates an occupancy rate of 11/12. For each case a single simulation run was undertaken.

Occupancy rate	Mean	Variance	Skewness	Kurtosis
2/3	3.579 (3.0)	25.99 (6)	4.596 (2.041)	42.61 (6.167)
5/6	13.97 (6.0)	1591 (30)	10.73 (2.008)	166.9 (6.033)
11/12	59.23 (12.0)	24399 (132)	6.234 (2.002)	58.10 (6.008)

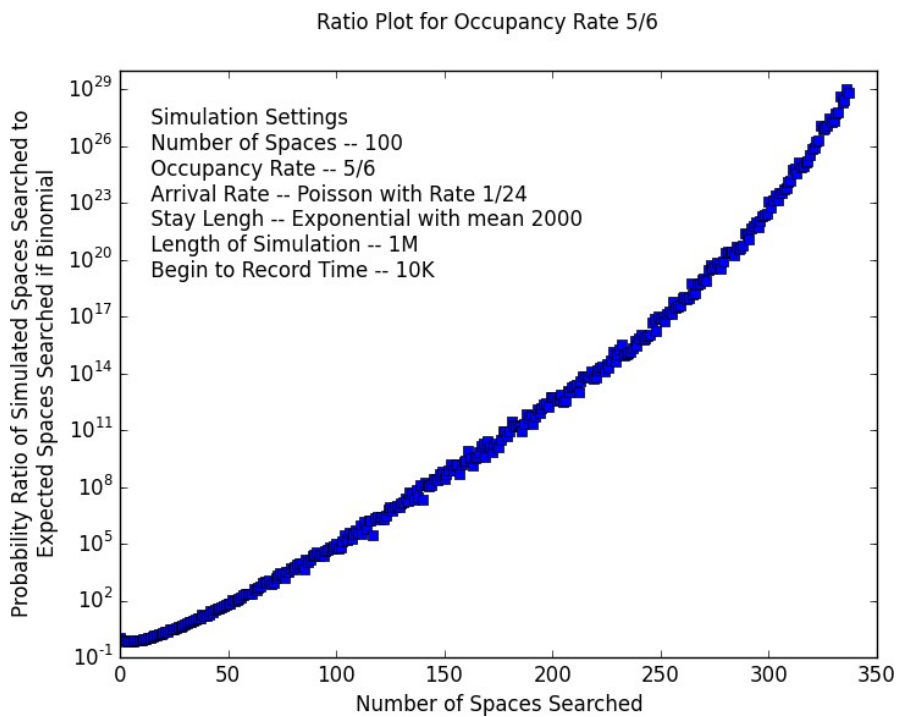
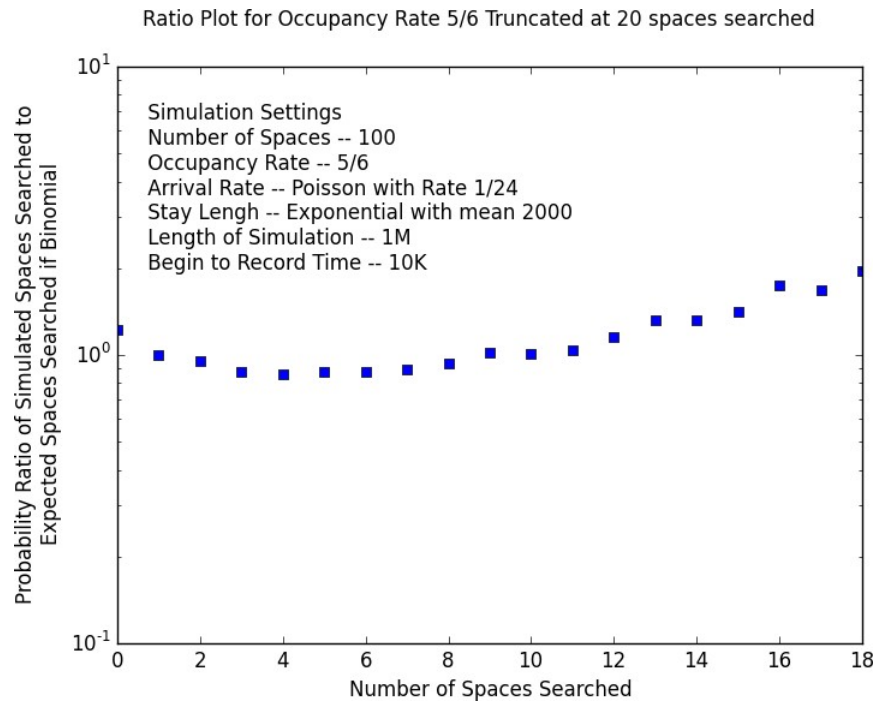
Table 2: Moments of the probability distribution of the number of parking spaces searched up to and including the first success, simulated and binomial approximation (in brackets)

Notes:

1. For the binomial distribution, the formula for the mean, variance, skewness, and kurtosis are $1/(1 - q)$, $q/(1 - q)^2$, $(1 + q)/q^{1/2}$, $6 + (1 - q)^2/q$, where q is the occupancy rate (wikipedia. Geometric distribution).

Four features of the results stand out. First, the ratio of the simulated mean of the number of parking spaces searched to the corresponding mean computed under the binomial approximation increases with the occupancy rate. Second, the simulated mean of the number of parking spaces searched increases approximately as the reciprocal of the vacancy rate squared (so that a halving of the vacancy rate results in approximately a quadrupling of the simulated mean). Third, the variance appears to increase somewhat faster than the reciprocal of the vacancy rate to the fourth power; that the standard deviation increases faster than the mean suggests that the tails become fatter as the vacancy rate increases, a result that can be confirmed in Figure 4 below, which presents the ratio plots. Fifth, in the simulations the skewness and the kurtosis are not monotonic in the occupancy rate. Since we undertook only one simulation run for each case, we strongly suspect that this is a statistical artifact, in particular that, in the second simulation

run, there was an extended period of time during which parking became almost completely blocked. Recall that the length of each simulation is 15,000 hrs. Even over



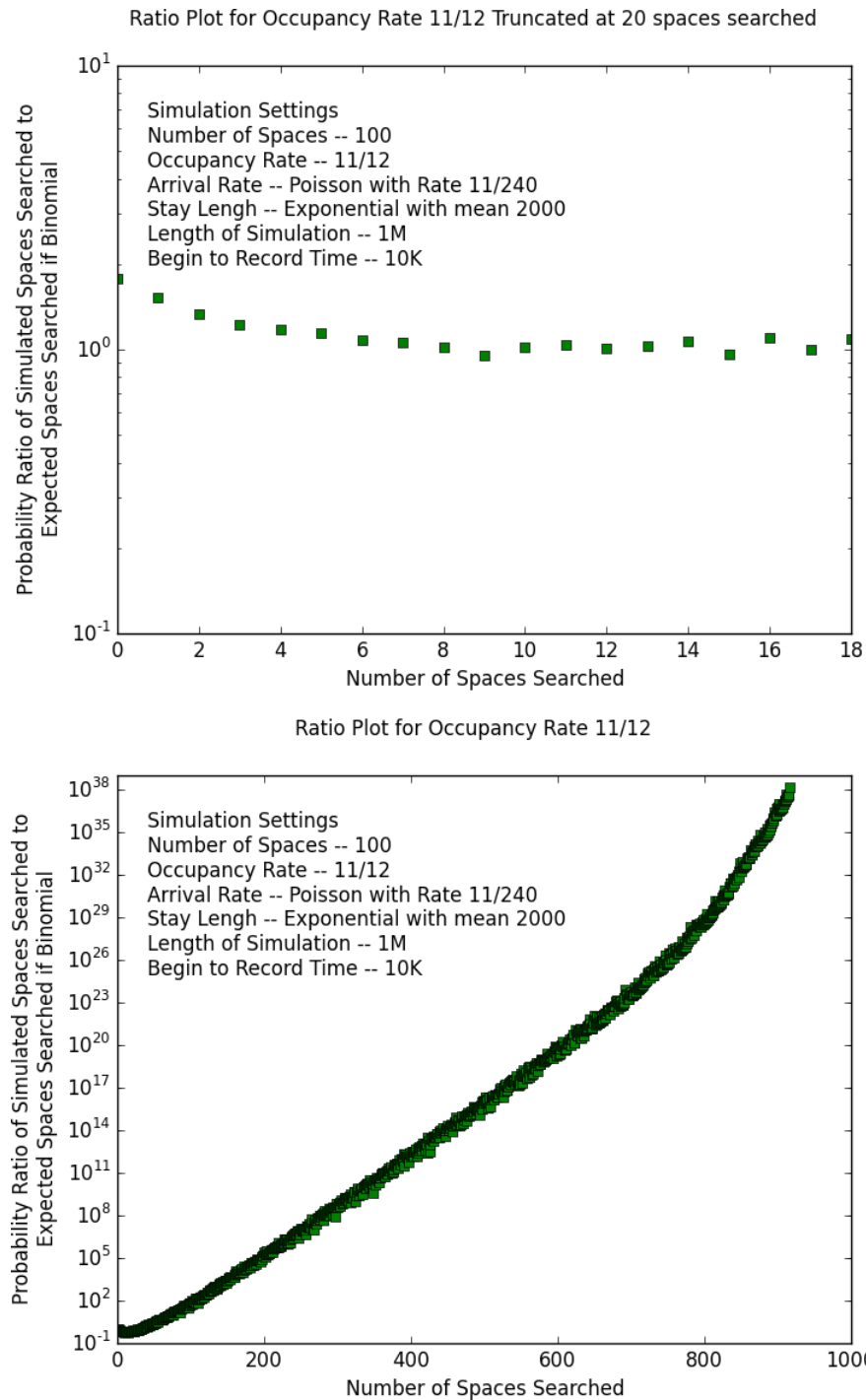


Figure 4: The ratio of the simulated probability that the number of parking spaces searched equals x to the corresponding probability under the binomial assumption. Truncated at 20 spaces searched

Panel A: occupancy rate = 5/6 Panel B: occupancy rate = 11/12
 Un-truncated figure
 Panel C: occupancy rate = 5/6 Panel D: occupancy rate = 11/12

such a long simulation time, it appears that the impact of anomalous extended periods with an exceptionally low vacancy rate can have a significant effect on the computed moments of the distribution.

Figure 4 is analogous to Figure 2, providing ratio plots for occupancy rates of $5/6$ (Panel A) and $11/12$ (Panel B). Panel A shows the same fat tail for the occupancy rate of $5/6$ as did Figure 2 for the occupancy rate of $2/3$. In the simulation, the largest number of parking spaces searched was close to 350, which corresponds to a search time of about 10.5 minutes¹¹. Panel B shows the same fat tail for the occupancy rate of $11/12$. In the simulation, the largest number of parking spaces searched was about 950, which corresponds to a search time of about 28.5 minutes.

Figure 5 is analogous to Figure 3, providing a space-time diagram for the evolution of occupied and vacant parking spaces for an occupancy rate of $5/6$. As in Figure 3, which is for an occupancy rate of $2/3$, bunching is very evident to the naked eye. In future work, we plan to investigate how the realized occupancy rate around the circle evolves over time. Intuitively, as the number of parking spaces becomes very large, the realized occupancy rate should converge to the expected occupancy rate. Thus, the variance of the realized occupancy rate should fall as the number of parking spaces around the circle increases.

¹¹ Since there are only 100 parking spaces around the circle, this entails traveling round the circle 3.5 times. We had intended that traveling round the circle would be a very rare event, but the CDF plot (not shown) indicates that the probability is in fact around 2%. In future simulation runs, we intend to increase the number of parking places round the circle so that this is less likely to occur.

Snapshots of the street every 1000 time steps with occupancy rate 5/6
 White is occupied Black is vacant

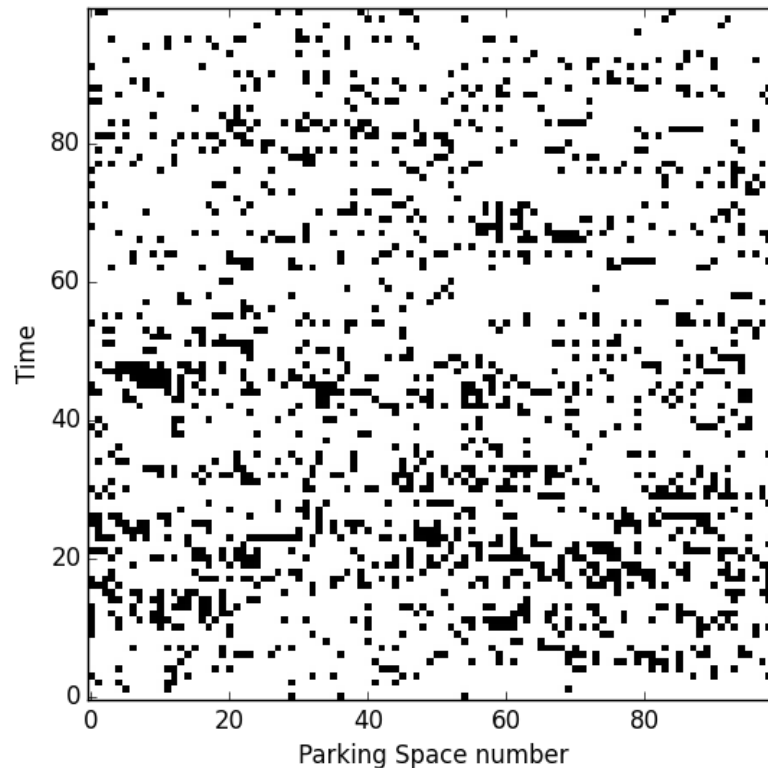


Figure 5: Bunching: Evolution of occupied and vacant parking spaces in space-time (occupancy rate = 5/6)

Notes:

1. The x-axis is the spatial axis, with each number indexing a particular parking space. Moving vertically describes the occupancy/vacancy pattern over time of a particular parking space. The y-axis is the time axis, with each cell providing a snapshot every 1000 time steps or units, or one half hour. Moving horizontally describes the occupancy/vacancy pattern over space at a particular point in time.
2. The parameters, including the begin-to-record time, are the same as in those listed in Figure 4, Panel A.
3. Since each time step corresponds to one-half hour, the Figure describes the evolution of parking around the circle over a 50 hr period.

3.2 *Doubling the entry rate and halving the expected stay length (parking duration) relative to the base case.*

Since the expected occupancy rate equals the Poisson entry rate times the expected stay length, doubling the entry rate and halving the expected stay length leaves the expected occupancy rate unchanged. Thus, this section examines the case where the turnover rate doubles but the expected occupancy rate remains unchanged. Before undertaking the simulation, we conjectured that doubling the turnover rate, while holding the expected occupancy rate unchanged, would reduce the mean number of parking spaces searched prior to finding a vacant parking space. We reasoned that the mean number of parking spaces searched would be sensitive to the number of long strings of occupied parking spaces, and that long strings of occupied parking spaces would break down more quickly, the higher the parking turnover rate. Indeed, in the limit, holding the occupancy rate fixed, as the parking duration approaches zero in the limit, and the entry rate approaches infinity, the law of large numbers should result in bunches of occupied parking spaces that exceed the mean number of parking spaces searched dissipating almost immediately.

It appears that our conjecture was incorrect. Table 3 shows the moments of the simulated distribution of the number of parking spaces searched up for the base case (the first row) and for the case where, holding constant the occupancy rate, the turnover rate is doubled (the second row). While the simulation results are subject to statistical error, which apparently may be significant despite the largest size, they suggest that a doubling of the turnover rate: i) actually increases the expected number of parking spaces searched, albeit only slightly; and ii) decreases the higher moments.

Exp. parking duration	Mean	Variance	Skewness	Kurtosis
2000 time units (base case)	3.579	25.99	4.596	42.61
1000 time units	3.692	24.24	3.238	16.34

Table 3: Probability distribution of the number of parking spaces searched, calculated through simulation. Occupancy rate = $2/3$. Row 1: Expected parking duration = 2000 time units (1 hr). Row 2: Expected parking duration = 1000 time units (1/2 hr).

The corresponding ratio plots, shown in Figure 6, display the same results from a somewhat different perspective. Apparently (the qualification again being statistical error) doubling the turnover rate increases the probability of the number of spaces searched being between 6 and 40, but decreases the probability of the number of spaces being above 40. This suggests that increasing the turnover rate increases the number of "very large" bunches but decreases the number of "extremely large bunches". We have no explanation for these unexpected results; they merit further examination.

3.3 Further Simulation Work Planned

On one hand, in the vast majority of situations, cars cruise for parking in two dimensions rather than in one dimension. As well, cruising for parking in two dimensions is *qualitatively* different from cruising in one dimension since in two dimensions but not in one dimension the individual driver has a choice between cruising for parking strategies. By itself, these considerations suggest that research effort should be allocated to exploring cruising for parking in two dimensions, which is the topic of the next phase of our research. On the other hand, "one has to walk before one can run". It is important to

understand cruising for parking in one dimension before moving on to the considerably more complicated situation of cruising for parking in two dimensions.

There are many simulation exercises that we could have done, but chose not to do, instead choosing to leave these for the two-dimensional case. One example is investigating the effects of alternative distributions of parking duration on the distribution of the number of parking spaces searched. Another example is balking. One of the outstanding issues in the microfoundations of parking search is how drivers choose between curbside and garage parking. Arnott and Rowse (2009) assume that drivers make this decision *ex ante*, before starting their trip. Realistically, however, since cruising for parking becomes increasingly frustrating with the time searched, even *ex ante* the rational driver may adopt the decision rule that he will park in a garage after a certain length of time in his curbside parking search. Furthermore, drivers acquire information relevant to their parking en route, and even when cruising for parking, acquire information en route relevant to the cruising-for-parking time distribution. As well, drivers differ in how they respond to the uncertainty concerning cruising-for-parking time. For all these reasons, some drivers can be expected to adopt a balking rule - park in a parking garage after a certain amount of time cruising for parking. Practically, balking is important in reducing the amount of cruising for parking.

Nevertheless, now that we have the results of the preliminary simulations of cruising for parking on a circle, we plan to undertake some additional simulations for this case.

When we designed our initial round of simulations, we had anticipated that running each

simulation for 15,000 hours of traffic would reduce "noise" to an acceptably low level. However, this appears not to be the case. "Outliers" -- observations in the extreme right tail of the distribution of cruising for parking times -- seem to affect the simulated distribution of cruising for parking times even for such long simulation runs. We conjecture that this occurs because, in these rare events, a stock of cars cruising for parking can accumulate so that any parking space that is vacated is taken almost immediately, with the result that it takes a very long time for curbside parking gridlock to unlock. One way to deal with this problem is to undertake even longer simulation runs. Another is to undertake multiple simulation runs of this length to investigate the distribution of the moments of the simulated distribution. Since noise becomes more important the higher the occupancy rate, it seems sensible to have longer simulation runs the higher the occupancy rate.

In hindsight, we made an error of judgment in our choice of how many parking spaces to have around the circle. The smaller the number of parking spaces, the more likely is parking to become gridlocked or close to gridlocked, so that noise becomes more important, and the more persistent are the effects of stochastic fluctuations. A sophisticated way to proceed would be to undertake two rounds of simulations, in the spirit of optimal sample design. The first round of simulations would estimate the probability distribution of the moments of the probability distribution of parking spaces searched, as a function of the length of the run, the number of parking spaces round the circle, and the occupancy rate. With this information, the second round of simulations

would be designed to maximize the statistical value of the information collected, given total amount of time devoted to simulation runs.

4. Discussion

The broad objective of this research project is to provide a conceptual basis for the determination of the optimal target curbside parking occupancy rate that would be applied in practice to set curbside meter rates. The problem is a difficult one. Account needs to be taken of traffic dynamics over the course of the day, which varies with location, heterogeneity in the street network and in drivers, the availability of off-street parking, and the interaction between curbside parking and traffic congestion.

In keeping with the spirit of microeconomics, we chose to strip the problem to its bare essentials, on the principle that one needs to understand the simplest case before adding "bells and whistles" in the direction of realism. In the first part of our research, we have chosen to analyze cruising for parking round a symmetric circle (an isotropic space), with identical drivers, in stochastic steady state, with no interaction between cars in traffic and cars cruising for parking.

The first paper on this research project, Arnott (2014), presented the conceptual basis for determining the optimal target curbside vacancy rate in a very similar¹² model, which could be used to numerically calculate the optimal rate, as a function of exogenous

¹² It differs in that each driver has a randomly-located destination around the circle that is independent of where he entered the circle. Furthermore, schedule delay costs are considered.

parameters and functions, specifically the values of travel time, time early and time late, visit length, regular driving speed, speed when cruising for parking, walking speed, and *the probability distribution of the number of parking spaces searched*, conditional on the expected curbside parking occupancy rate.

The objective of this phase of the research was to solve for the probability distribution of the number of parking spaces searched, conditional on the curbside parking occupancy rate. At first, we considered solving this problem analytically. We abandoned this line of attack. The "proper" problem appeared intractable due to the high dimension of the state space (the state of the system is described by not only the occupancy status of each of the parking spaces, but also the number and location of cars cruising for parking and the remaining parking time at each occupied parking space¹³ (except for the negative exponential distribution for parking duration). Perhaps we could have made analytical progress through approximating assumptions, but to determine the accuracy of any approximation would require knowing the proper distribution. We therefore proceeded to simulation.

We had mixed success in our simulation work. On the good side, taken together the simulation results provided conclusive evidence that, contrary to a standard assumption in the literature, holding fixed the occupancy rate the probability distribution of the number of parking spaces searched is not well approximated by the geometric distribution. The

¹³ Except for the negative exponential distribution, which we assumed, for which, from the perspective of an outside observer, the expected duration of time until a parking space is vacated is independent of the duration of time already parked.

true distribution has fatter tails than the geometric distribution, resulting in a higher expected number of parking spaces searched. Due to both temporal and spatial autocorrelation generated by the true data-generating process, there is more bunching of occupied parking spaces than would occur under the geometric distribution, which is derived on the assumption of zero temporal and spatial autocorrelation. On the bad side, even though each run simulated 15,000 hours of parking round the circle, it appears that there is still significant uncertainty concerning the moments of the true distribution. The intuitive reason seems to be that cruising on a circle of finite length generates rare episodes where parking approaches gridlock, and that even though these episodes are rare they generate such high numbers of expected parking spaces searched and extend for sufficiently long periods of time that they affect the form of the simulated distribution. The problem is remediable by extending the run time of each simulation, running multiple simulations, and increasing the number of parking spaces round the circle. Once acceptably tight bounds have been determined for the moments of the distribution, as a function of the exogenous sets of parameters employed, it will be possible to run a regression with the simulated moments of the distribution as the dependent variables and the exogenous parameters as the independent variables, and to use the regression results to estimate the moments of the distribution for other sets of parameters.

The research reported here ignores a phenomenon considered in Arnott (2014) that is potentially important at high occupancy rates. When cruising-for-parking times are significant, many drivers select parking spaces that are far from their destinations. The duration of time a person parks equals the visit duration plus the time it takes her to walk

from the parking space to her destination and back again. Unless the visit duration is chosen to exactly or more than offset the walking duration, which is unrealistic, an increase in the *expected* occupancy rate increases the expected parking duration. This effect *per se* does not affect the distribution of cruising-for-parking times corresponding to the occupancy. However, holding constant the entry rate, an increase in the expected parking duration causes the expected occupancy rate to increase, which does affect the distribution of cruising-for-parking times. Furthermore, for the same reason a *random* increase in the realized occupancy rate increases expected parking duration, which has a positive feedback on the realized occupancy rate. Thus, the endogeneity of parking duration amplifies fluctuations in the occupancy rate. These effects would be captured in a simulation of cruising for parking based on the model in Arnott (2014), which enriches the model of this paper to include schedule delay and the endogeneity of both the probability distribution of parking duration and the occupancy rate.

Our research has focused on cruising for parking round a circle. In the next phase of our research on this project, we shall consider cruising for parking on a Manhattan network. All of the issues that arise in considering cruising for parking round a circle arise as well on a Manhattan network. But there are two important complications. The first is obvious, and was commented on earlier. The state-time analysis of parking occupancy needs to be extended from two to three dimensions. The second is that each driver must choose a cruising for parking strategy on the Manhattan network. How does a driver acquire information on the parking state of the system? How does he forecast the future state of the system? In light of his information and his forecast, how does he choose his

cruising-for-parking strategy¹⁴? In what ways does the parking state of the system, including the occupancy rate, depend on the cruising-for-parking strategies that drivers adopt?

A general issue that arises in modeling downtown parking and traffic congestion is how to treat stochasticity. Let us simplify by considering only the stochastic steady state. There is a large literature on traffic signal control. The classic models in this literature apply stochastic queuing theory to individual intersections in isolation. Stochasticity is essential to these models. Suppose, to the contrary, that the arrival rate is deterministic. If the arrival rate is greater than the service rate of the intersection, the queue at the intersection would grow without limit. If the arrival rate is less than the service rate, there would be no queues. But individual intersections do not operate in isolation. Instead, there is a network of linked intersections. Suppose that, just by chance, an abnormally high number of cars enter a link from either curbside or off-street parking spaces on that link in the red stage of the downstream signal cycle, resulting in an abnormally long queue at the traffic signal, which takes more than one cycle length to service. This stochastic fluctuation will propagate through the network, with the next downstream link receiving an abnormally high number of cars on the next signal cycle, and also perhaps the links to the right and the left of that signal. Since the stochastic

¹⁴ A complication is that a pure strategy equilibrium in the rational choice of cruising-for-parking strategy may not exist. Suppose that all drivers but one adopt a common cruising-for-parking strategy. It may then be optimal for the remaining driver to adopt a different strategy. In this situation, conditional on an equilibrium existing, it will be mixed strategy equilibrium.

fluctuation is not serviced on a single signal cycle, its impact will persist to the next signal cycle, though it will be attenuated.

Thus, the network of downtown traffic intersections is a set of linked servers, so that random fluctuations propagate through the network. We noted earlier that curbside parking spaces too may be viewed as a network of linked servers, though the queuing mechanism is different from that of intersections. The two networks of servers share some common qualitative characteristics. Importantly, bunching is important on both networks. The two networks also interact since both cars in transit and cars cruising for parking contribute to traffic density. Thus, it will be fruitful to explore the properties of simulation models that treat queuing at intersections, cruising for parking, and traffic congestion simultaneously.

In a seminal contribution, Geroliminis and Daganzo (2007) showed empirically for a neighborhood of Yokohama that the *aggregate* behavior of traffic is well represented by a deterministic macroscopic fundamental diagram. Through both empirical observation and simulation, subsequent work has shown that the same is true for neighborhoods in other cities, although how tight the fit of observed flow-density pairs over the course of the day is to the diagram depends on, *inter alia*, the similarity of streets in the neighborhood. Put alternatively, though stochasticity is important at the level of the individual intersection and on the individual link, the aggregate behavior of the network of intersections, joined by congestible links, is almost deterministic. Does the same qualitative result hold for curbside parking? Our simulations indicated that noise is

important for cruising for parking around a circle with 100 curbside parking spaces. Is this noise of secondary importance in describing the aggregate behavior of cruising for parking at the level of a downtown neighborhood? We shall investigate this question in our subsequent work that deals with curbside parking on a Manhattan network. Since no reliable method has yet been developed to distinguish between cars cruising for parking versus cars in transit, empirically the most promising approach would appear to be to estimate fundamental diagrams for all traffic in a variety of neighborhoods, including the density of curbside parking spaces and the curbside parking occupancy rate as explanatory variables.

Our work thus far ignores that cruising for parking affects traffic congestion, which is the focus of Arnott and Inci (2006). The third phase of this project too will ignore the interaction between cruising for parking and traffic congestion. Geroliminis (2015) reports the results of a simulation model that accounts for this interaction, but under the binomial approximation.

5. *Conclusion*

This report describes the work undertaken in the second stage of the research project, "Towards Inferring Welfare Changes from Changes in Curbside Occupancy Rates: A Theoretical Analysis Motivated by SFpark and LA Express Park", Department of Transportation Contract No. 65A0528, funding for which is shared with Caltrans. The first stage of the research, which was completed before the funds for the project were released, presented a model of cruising for parking around a circle, and for its stochastic

steady state related welfare to the form of the demand function, the curbside parking occupancy rate, and *the probability distribution of the number of curbside parking spaces searched before finding a vacant parking space*. It therefore provided the conceptual basis for determining the optimal (*ex ante*) target curbside occupancy rate in stochastic steady state. The results are published in Arnott (2014).

This second stage of the project investigated the stochastic steady state probability distribution of the number of curbside parking spaces searched before finding a vacant parking space, with a particular focus on how this distribution changes with the average (*ex ante*) occupancy rate. It did this primarily through simulation.

All previous work on this problem, both theoretical (Arnott and Rowse, 1999; Anderson and de Palma, 2004) and simulation (Geroliminis, 2015), made the approximating assumption that occupied parking spaces are generated by a spatially and temporally invariant Poisson process with a rate equal to the occupancy rate -- intuitively, the probability that a particular space is occupied equals the occupancy rate, independent of both the occupancy history of that parking space and the occupancy status of neighboring parking spaces. We termed this the *binomial approximation* after the binomial distribution, which is derived from this assumption. Thus, for example, with a curbside parking occupancy rate of two-thirds, a car cruising for parking is viewed as taking independent draws with replacement from an urn in which two-thirds of the balls are marked occupied and one-third vacant. Previous simulation work by Levy *et al.* (2012) had strongly suggested that this approximating assumption is an increasingly poor one as

the occupancy rate increases, and substantially underestimates the expected number of spaces searched for occupancy rates exceeding 85%.

Each of our simulations traced the occupancy history of 100 curbside parking spaces equally spaced around a circle for 15,000 hours, across simulations varying the time- and location-independent entry rate of cars and their expected parking durations so as to achieve a preselected average occupancy rate.

The principal finding of our research is that the binomial approximation is an increasingly bad one as the curbside parking occupancy rate increases. In the base case, the mean parking duration was one hour. If the entry rate is chosen so as to achieve an occupancy rate of $2/3$, under the binomial approximation the expected number of curbside parking spaces searched is 3.0 while in the simulation run it is 3.579. With this occupancy rate, therefore, the expected number of curbside parking spaces searched is underestimated by about 20% ((realized minus approximated) divided by approximated). The degree of underestimation increases sharply as the occupancy rate approaches 100%. With an occupancy rate of $11/12$, the degree of underestimation is 500%; under the binomial approximation the expected number of curbside parking spaces searched is 12.0, while in the simulation it is 59.23. The degree of underestimation of the standard deviation behaves similarly. The proximate reason for the discrepancy between the two distributions is that the simulated distributions have much fatter tails than the approximating distribution. These fatter tails derive from the bunching of occupied

parking spaces, which in turn derive from the positive autocorrelation of occupied parking spaces in space-time.

In designing our research, we had thought that simulating 15000 hours of traffic on the circle would allow us to estimate the moments of the simulated distribution to a high degree of accuracy. However, this appears not to be the case. It seems that there are episodes in which parking around the circle becomes almost gridlocked, and that when this occurs the almost-gridlock takes a long time to unlock. While these episodes are rare -- less rare the higher the occupancy rate -- they significantly affect the estimated distribution. We conjecture that these episodes would be even rarer, and less quantitatively important, on a circle with more parking spaces.

The optimal target curbside parking occupancy rate proposed by Shoup (2006) and the target curbside parking occupancy rates implemented in SFpark are averages over time and space (if applied to more than one block). Since the number of expected number of curbside parking spaces searched is convex in the realized local occupancy rate, the more variable is the realized local occupancy rate, conditional on an average occupancy rate, the larger the expected number of curbside parking spaces searched. There are three sources of variability in the realized local occupancy rate: i) variation in the average occupancy rate across locations that are subject to a common target occupancy rate; ii) variation in the average occupancy rate over time within time periods that are subject to a common target occupancy rate; and iii) stochastic variation. In our research, we considered only stochastic variation. If the two other sources of variation were taken into

account too, holding constant the average occupancy rate, the expected number of curbside parking spaces searched would normally be larger, perhaps considerably larger, than obtained in our simulations. Thus, if use of the binomial assumption seriously underestimates the number of parking spaces searched when only stochastic variation is taken into account, it even more seriously underestimates the number of parking spaces searched when the other two sources of variation are accounted for.

The main conclusion of our work is therefore that *the binomial approximation should be avoided in theoretical and simulation research on cruising for parking.*

Where to from here? At first glance, it might seem that the way to go is to collect real-world data relating the distribution of the number of parking spaces searched to the average occupancy rate and to the realized local occupancy rate. The critical weakness with this approach is that one cannot distinguish between cars that are cruising for parking from cars in transit. With cell phone data, one could in principle trace backward in time the history of a downtown parker's journey, albeit with noise. One would identify the parker's destination by the location where he stays for an extended period of time after parking, and his parking location by where he stops and parks his car. Given the distribution of distances between the destination and parking locations, one might be able to infer both the "average" cruising-for-parking strategy and the expected number of curbside parking spaces searched before finding a vacant space. But cell phone data are not available, and the theory relating the distribution of distances between the destination

and parking locations to the expected number of curbside parking spaces searched has yet to be developed.

Thus, the more promising avenue appears to be *a priori* theoretical analysis. Since a pure analytical approach appears to be intractable, we believe that the most promising approach is to obtain estimates of the distribution of parking search times from simulations based on theoretical models. In this paper, we employed this strategy, basing our simulations on a theoretical model of cars cruising for parking round a circle in stochastic steady state. In the third and final phase of our research on this project, we shall employ the same strategy, but base our simulations on a theoretical model of cars cruising for parking on a Manhattan grid network in stochastic steady state. The grid model is qualitatively different from the circular model. In the circular model, a driver has no decisions to make; he just drives around the circle until he encounters his first vacant parking space. In the grid model, in contrast, a driver decides on a cruising-for-parking strategy, trading off expected cruising for parking time against expected walking time. If the average occupancy rate is low, his optimal strategy might be simply to cruise around his destination block until he encounters a vacant space. If he drives all the way round the block without finding a space, he may choose to drive around the block again, figuring that since the average occupancy rate is low, he is likely to find a space that has recently been vacated. If the average occupancy rate is high, a naive driver might continue circling his destination block, figuring (incorrectly) that if all the parking spaces around his destination block are taken, so too are all the parking spaces around adjacent blocks, and that he is better off just waiting for a space around his destination block to

open up. A sophisticated driver however would take into account the pattern of spatial autocorrelation of occupied parking spaces.

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TECHNICAL APPENDIX

The Simulation Algorithm

This appendix provides a purely verbal description of the simulation algorithm and then a verbal flow chart description of the simulation. At the very end is a link to the actual code that was employed.

Description of the Algorithm

Given the arrival rate, generate a list of arrival times less than the total running time. Create a car object for every arrival time, and uniformly pick a starting location on the block. Initialize all parking spaces as occupied with exponentially distributed times until they are no longer occupied. Begin the simulation, for every time step less than the total running time, we do the following, in the following order: Allow spaces to empty, if a car was at a space that is empty, it may pull in and occupy it, and a new departure time for the parking space is set chosen from the occupancy time exponential distribution, the car is removed from the road. If the current time step when this occurred is past the threshold to record, record the length of time this car was searching. All cars that are at spaces that are occupied advance one space modulo the length of our block.

A More Flow-oriented Description

Length of block = n
 Arrival rate = λ
 Stay length = S
 Length of simulation = T
 Record after time = T_0

while($t < T$)
 allow spaces to empty

 cars pull onto the road

 If a car is next to a space that is empty, let it pull in. In cases of a tie the space goes to who arrived at it first

 If $t > T_0$

 record the difference in time (t - starting time for this car)

 advance all cars one space mod n

$t = t+1$

Hyperlink <https://github.com/ParkerWilliams/Cruising-For-Parking-1d>